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Use a pen!

01.05.2015

General Relativity - Exercise Sheet 3

Σ	1	2	3	4	5
	10	8	8.5	10	3

Problem 1 (Fun with Indices) [10 points]

Consider the metric $g^{\mu\nu}$ with $\text{diag}(\{g^{\mu\nu}\}_{\mu,\nu\in\{0,1,2,3\}}) = (a, b, -b^{-1}, \tan(c))$.

a) Calculate $g_{\mu\nu}$.

As stated in lecture 5, $g_{\mu\nu}$ is simply the inverse of $g^{\mu\nu}$. Since $g^{\mu\nu}$ is diagonal, its inverse can be computed by inverting all entries.

$$g_{\mu\nu} = (g^{\mu\nu})^{-1} \implies \text{diag}(\{g_{\mu\nu}\}_{\mu,\nu\in\{0,1,2,3\}}) = \left(\frac{1}{a}, \frac{1}{b}, -b, \cot(c)\right) \quad ✓$$

b) Let's assume that we're in Cartesian coordinates (t, x, y, z). What is the line element ds^2?

$$ds^2 = ds \cdot ds = ds_\mu ds^\mu = g_{\mu\nu} ds^\mu ds^\nu = \frac{1}{a} dt^2 + \frac{1}{b} dx^2 - b\, dy^2 + \cot(c)\, dz^2 \quad ✓$$

c) Given $\{x^\mu\}_{\mu\in\{0,1,2,3\}} = (1, 0, 1, 0)^T$, what is x_μ?

$$\{x_\mu\}_{\mu\in\{0,1,2,3\}} = \{g_{\mu\nu}\}_{\mu,\nu\in\{0,1,2,3\}} \{x^\nu\}_{\nu\in\{0,1,2,3\}} = \left(\frac{1}{a}, 0, -b, 0\right) \quad ✓$$

d) You're given the line element

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2.$$

What sort of coordinate system are we using? What does the metric look like?

There is no unique answer to the above questions. Two obvious possibilities are

$$ds = (c\,dt, dr, r\,d\phi)^T \quad \text{and} \quad ds = (dt, dr, d\phi)^T,$$

in which cases the metric is given by

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \quad ✓$$

respectively. There are more.

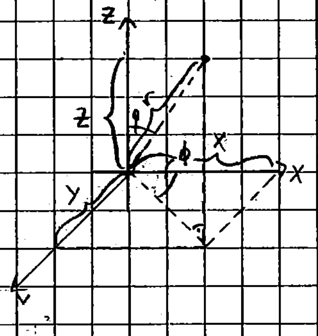
Problem 2 (Metric tensor and equations of motion) [10 points]

The metric can be written as

$$g_{\mu\nu} = \eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}}.$$

In three-dimensional Euclidean space, this is just a coordinate transformation.

a) Find $g_{\mu\nu}$ for polar coordinates (r, θ, ϕ). Neglect the zero-components, i.e. $\mu = i \in I = \{1,2,3\}$.



The transformation to polar coordinates is given by

$$x = x(r,\theta,\phi) = r\cos\phi\,\sin\theta, \quad y = y(r,\theta,\phi) = r\sin\phi\,\sin\theta$$

$$z = z(r,\theta) = r\cos\theta, \quad \text{where } r \in \mathbb{R}_+,\ \theta \in [0,\pi],\ \phi \in [0, 2\pi]$$

To calculate the with the above formula, we need the Jacobian J of this transformation

$$J = \{J^k{}_i\}_{k,i\in I} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$\{g_{ij}\}_{i,j\in I} = \underbrace{\{\eta_{kl}\}_{k,l\in I}}_{-\mathbb{1}_3} \{J^k{}_i\}_{k,i\in I} \{J^l{}_j\}_{l,j\in I} = -J^T J$$

$$= -\begin{pmatrix} \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta \\ -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} \cos^2\phi\sin^2\theta + \sin^2\phi\sin^2\theta + \cos^2\theta & r\cos^2\phi\sin\theta\cos\theta + r\sin^2\phi\sin\theta\cos\theta - r\sin\theta\cos\theta \\ r\cos^2\phi\sin\theta\cos\theta + r\sin^2\phi\sin\theta\cos\theta - r\sin\theta\cos\theta & r^2\cos^2\phi\cos^2\theta + r^2\sin^2\phi\cos^2\theta + r^2\sin^2\theta \\ -r\sin\phi\cos\phi\sin^2\theta + r\sin\phi\cos\phi\sin^2\theta & -r^2\sin\phi\cos\phi\sin\theta\cos\theta + r^2\sin\phi\cos\phi\sin\theta\cos\theta \\ -r\sin\phi\cos\sin^2\theta + r\sin\phi\cos\phi\sin^2\theta & \\ -r^2\sin\phi\cos\phi\sin\theta\cos\theta + r^2\sin\phi\cos\phi\sin\theta\cos\theta & \\ r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi\sin^2\theta & \end{pmatrix} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

✓

01.05.2014

b) Now limit yourself to the surface of a sphere in $\mathbb{R}^3$, i.e. a 2-sphere. Calculate all remaining Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$ using

$$\Gamma^\alpha{}_{\mu\nu} = \frac{g^{\alpha\rho}}{2}\left(g_{\nu\rho,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}\right)$$

Remark: In the above formula, a shorthand notation is employed, in which nabla symbols and partial derivatives are dropped and instead the indices with respect to which a quantity is derivated are separated by semi-colons and commas, respectively. E.g. $g_{\nu\rho,\mu}$ stands for $\frac{\partial g_{\nu\rho}}{\partial x^\mu}$. ✓

2D, surface of sphere in $\mathbb{R}^3$

On a three-dimensional manifold, there are a total of $n^3 = 3^3 = 27$ Christoffel symbols. However, since $\Gamma^\alpha{}_{\mu\nu}$ is symmetric in its lower indices, i.e. $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$, a total of 9 symbols occur twice so that we only need to calculate 18 Christoffel symbols. In $\mathbb{R}^3$, we write the formula for $\Gamma^\alpha{}_{\mu\nu}$ as

$$\Gamma^a{}_{ij} = \frac{g^{ak}}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right), \text{ where } a,i,j,k \in I = \{1,2,3\} \text{ and } x^i = \begin{cases} r & \text{for } i=1 \\ \theta & \text{for } i=2 \\ \phi & \text{for } i=3 \end{cases}$$

With this preparation, we calculate the following symbols (* when limited to sphere, r = const)

there's no 1 = r component

$$\Gamma^{1}{}_{13} = \frac{1}{2} g^{1k}\left(\frac{\partial g_{1k}}{\partial x^1} + \frac{\partial g_{1k}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^k}\right) = g^{1k}\underbrace{\frac{\partial g_{1k}}{\partial x^1}}_{0 \text{ for all } k} - \frac{1}{2} g^{1k}\underbrace{\frac{\partial g_{11}}{\partial x^k}}_{0 \text{ for all } k} = 0$$

$$\Gamma^{1}{}_{12} = \frac{1}{2} g^{1k}\left(\frac{\partial g_{2k}}{\partial x^1} + \underbrace{\frac{\partial g_{1k}}{\partial x^2}}_{0} - \underbrace{\frac{\partial g_{12}}{\partial x^k}}_{0}\right) = \frac{1}{2} g^{12}\frac{\partial g_{22}}{\partial x^1} = \frac{1}{2}\cdot 0 \cdot \frac{\partial r^2}{\partial r} = 0 \cdot r = 0 = \Gamma^{1}{}_{21}$$

$$\Gamma^{1}{}_{13} = \frac{1}{2} g^{1k}\left(\frac{\partial g_{3k}}{\partial x^1} + \underbrace{\frac{\partial g_{1k}}{\partial x^3}}_{0} - \underbrace{\frac{\partial g_{13}}{\partial x^k}}_{0}\right) = \frac{1}{2} g^{13}\frac{\partial g_{33}}{\partial x^1} = \frac{1}{2}\cdot 0 \cdot \frac{\partial r^2\sin^2\theta}{\partial r} = 0 = \Gamma^{1}{}_{31}$$

$$\Gamma^{1}{}_{22} = \frac{1}{2} g^{1k}\left(\underbrace{\frac{\partial g_{2k}}{\partial x^2}}_{0} + \underbrace{\frac{\partial g_{2k}}{\partial x^2}}_{0} - \frac{\partial g_{22}}{\partial x^k}\right) = \frac{1}{2} g^{11}\left(-\frac{\partial g_{22}}{\partial x^1}\right) = -\frac{1}{2}(-1)\frac{\partial r^2}{\partial r} = r \quad \left(\Gamma^1{}_{22} \overset{*}{=} 0\right)$$

$$\Gamma^{1}{}_{23} = \frac{1}{2} g^{1k}\left(\frac{\partial g_{3k}}{\partial x^2} + \underbrace{\frac{\partial g_{2k}}{\partial x^3}}_{0} - \underbrace{\frac{\partial g_{23}}{\partial x^k}}_{0}\right) = \frac{1}{2} g^{13}\frac{\partial g_{33}}{\partial x^2} = \frac{1}{2}\cdot 0 \cdot \frac{\partial r^2\sin^2\theta}{\partial \theta} = 0 = \Gamma^{1}{}_{32}$$

$$\Gamma^{1}{}_{33} = \frac{1}{2} g^{1k}\left(\underbrace{\frac{\partial g_{3k}}{\partial x^3}}_{0} + \underbrace{\frac{\partial g_{3k}}{\partial x^3}}_{0} - \frac{\partial g_{33}}{\partial x^k}\right) = -\frac{1}{2}\left(g^{11}\frac{\partial g_{33}}{\partial x^1} + \underbrace{g^{12}}_{0}\frac{\partial g_{33}}{\partial x^2}\right) = -\frac{1}{2}(-1)\frac{\partial r^2\sin^2\theta}{\partial r} = r\sin^2\theta$$

$$\left(\Gamma^1{}_{33} \overset{*}{=} 0\right)$$

$$\Gamma^2_{11} = \frac{1}{2} g^{2k}\Big(\underbrace{\frac{\partial g_{1k}}{\partial x^1}}_{0} + \underbrace{\frac{\partial g_{1k}}{\partial x^1}}_{0} - \underbrace{\frac{\partial g_{11}}{\partial x^k}}_{0}\Big) = 0 \qquad \Gamma^2_{12} = \frac{1}{2} g^{2k}\Big(\frac{\partial g_{2k}}{\partial x^1} + \underbrace{\frac{\partial g_{1k}}{\partial x^2}}_{0} - \underbrace{\frac{\partial g_{12}}{\partial x^k}}_{0}\Big) = \frac{1}{2} g^{22}\frac{\partial g_{22}}{\partial x^1} = \frac{1}{r} = \Gamma^2_{21}$$

$$\big(\Gamma^2_{12} = \Gamma^2_{21} \overset{*}{=} 0\big)$$

$$\Gamma^2_{13} = \frac{1}{2} g^{2k}\Big(\frac{\partial g_{3k}}{\partial x^1} + \underbrace{\frac{\partial g_{1k}}{\partial x^3}}_{0} - \underbrace{\frac{\partial g_{13}}{\partial x^k}}_{0}\Big) = \frac{1}{2} g^{23}\frac{\partial g_{33}}{\partial x_1} = \frac{1}{2}\cdot 0 \cdot \frac{\partial r^2\sin^2\theta}{\partial r} = 0 = \Gamma^2_{31}$$

$$\Gamma^2_{22} = \frac{1}{2} g^{2k}\Big(\underbrace{\frac{\partial g_{2k}}{\partial x^2}}_{0} + \underbrace{\frac{\partial g_{2k}}{\partial x^2}}_{0} - \frac{\partial g_{22}}{\partial x^k}\Big) = \frac{1}{2} g^{21}\frac{\partial g_{22}}{\partial x^1} = \frac{1}{2}\cdot 0 \cdot \frac{\partial r^2}{\partial r} = 0$$

$$\Gamma^2_{23} = \frac{1}{2} g^{2k}\Big(\frac{\partial g_{3k}}{\partial x^2} + \underbrace{\frac{\partial g_{2k}}{\partial x^3}}_{0} - \underbrace{\frac{\partial g_{23}}{\partial x^k}}_{0}\Big) = \frac{1}{2} g^{23}\frac{\partial g_{33}}{\partial x^2} = \frac{1}{2}\cdot 0 \cdot \frac{\partial r^2\sin^2\theta}{\partial \theta} = 0 = \Gamma^2_{32}$$

$$\Gamma^2_{33} = \frac{1}{2} g^{2k}\Big(\underbrace{\frac{\partial g_{3k}}{\partial x^3}}_{0} + \underbrace{\frac{\partial g_{3k}}{\partial x^3}}_{0} - \frac{\partial g_{33}}{\partial x^k}\Big) = -\frac{1}{2}\Big(\underbrace{g^{21}}_{0}\frac{\partial g_{33}}{\partial x^1} + g^{22}\frac{\partial g_{33}}{\partial x^2}\Big) = -\frac{1}{2}\Big(\frac{1}{r^2}\Big)\frac{\partial (r^2\sin^2\theta)}{\partial\theta} = -\sin\theta\cos\theta$$

$$\Gamma^3_{11} = \frac{1}{2} g^{3k}\Big(\underbrace{\frac{\partial g_{1k}}{\partial x^1}}_{0} + \underbrace{\frac{\partial g_{1k}}{\partial x^1}}_{0} - \underbrace{\frac{\partial g_{11}}{\partial x^k}}_{0}\Big) = 0 \qquad \Gamma^3_{12} = \frac{1}{2} g^{3k}\Big(\frac{\partial g_{2k}}{\partial x^1} + \underbrace{\frac{\partial g_{1k}}{\partial x^2}}_{0} - \underbrace{\frac{\partial g_{12}}{\partial x^k}}_{0}\Big) = \frac{1}{2}\underbrace{g^{32}}_{0}\frac{\partial g_{22}}{\partial x^1} = 0 = \Gamma^3_{21}$$

$$\Gamma^3_{13} = \frac{1}{2} g^{3k}\Big(\frac{\partial g_{3k}}{\partial x^1} + \underbrace{\frac{\partial g_{1k}}{\partial x^3}}_{0} - \underbrace{\frac{\partial g_{13}}{\partial x^k}}_{0}\Big) = \frac{1}{2} g^{33}\frac{\partial g_{33}}{\partial x^1} = \frac{1}{2}\frac{1}{r^2\sin^2\theta}\frac{\partial r^2\sin^2\theta}{\partial r} = \frac{1}{r} = \Gamma^3_{31} \qquad \big(\Gamma^3_{13} = \Gamma^3_{31} \overset{*}{=} 0\big)$$

$$\Gamma^3_{22} = \frac{1}{2} g^{3k}\Big(\underbrace{\frac{\partial g_{2k}}{\partial x^2}}_{0} + \underbrace{\frac{\partial g_{2k}}{\partial x^2}}_{0} - \frac{\partial g_{22}}{\partial x^k}\Big) = -\frac{1}{2}\underbrace{g^{31}}_{0}\frac{\partial g_{22}}{\partial x^1} = 0$$

$$\Gamma^3_{23} = \frac{1}{2} g^{3k}\Big(\frac{\partial g_{3k}}{\partial x^2} + \underbrace{\frac{\partial g_{2k}}{\partial x^3}}_{0} - \underbrace{\frac{\partial g_{23}}{\partial x^k}}_{0}\Big) = \cot\theta = \Gamma^3_{32} \qquad \Gamma^3_{33} = \frac{1}{2} g^{3k}\Big(\underbrace{\frac{\partial g_{3k}}{\partial x^3}}_{0} + \underbrace{\frac{\partial g_{3k}}{\partial x^3}}_{0} - \frac{\partial g_{33}}{\partial x^k}\Big) = -\frac{1}{2} g^{33}\underbrace{\frac{\partial g_{33}}{\partial x^3}}_{0} = 0$$

c) Write down the equation of motion by using the geodesic equation.

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \qquad \big(\text{in } \mathbb{R}^3:\ \ddot{x}^i + \Gamma^i_{ab}\dot{x}^a\dot{x}^b = 0 \text{ with } i,a,b \in I = \{1,2,3\}\big) \qquad (1)$$

$$x^1 = r:\quad \ddot{x}^1 + \Gamma^1_{ab}\dot{x}^a\dot{x}^b = \ddot{x}^1 + \Gamma^1_{22}\dot{x}^2\dot{x}^2 + \Gamma^1_{33}\dot{x}^3\dot{x}^3 = \ddot{r} + r\dot{\theta}^2 + r\sin^2\theta\,\dot{\varphi}^2 = 0$$

$$x^2 = \theta:\quad \ddot{x}^2 + \Gamma^2_{ab}\dot{x}^a\dot{x}^b = \ddot{x}^2 + \Gamma^2_{12}\dot{x}^1\dot{x}^2 + \Gamma^2_{21}\dot{x}^2\dot{x}^1 + \Gamma^2_{33}\dot{x}^3\dot{x}^3 = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\,\dot{\varphi}^2 = 0$$

$$x^3 = \phi:\quad \ddot{x}^3 + \Gamma^3_{ab}\dot{x}^a\dot{x}^b = \ddot{x}^3 + \Gamma^3_{13}\dot{x}^1\dot{x}^3 + \Gamma^3_{31}\dot{x}^3\dot{x}^1 + \Gamma^3_{23}\dot{x}^2\dot{x}^3 + \Gamma^3_{32}\dot{x}^3\dot{x}^2 = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0$$

When limited to the surface of a sphere, the e.o.m.s simplify to

$$r:\ \ddot{r} = 0, \qquad \theta:\ \ddot{\theta} + \sin\theta\cos\theta\,\dot{\varphi}^2 = 0, \qquad \varphi:\ \ddot{\varphi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0 \qquad \checkmark$$

02.05.2015

Problem 3 (Weak field I: Equations of motion) [10 points]

Consider the weak-field limit of general relativity with a central potential $\Phi(r) = -\frac{1}{c^2}\frac{GM}{r}$. The line element reads

$$ds^2 = (1+\Phi(r))\,dt^2 - (1-2\Phi(r))\,(dr^2 + r^2 d\Omega^2), \quad \text{with } d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

a) Find the action $S = -mc\int d\tau \sqrt{\dot{x}_\mu \dot{x}^\mu}$

$$\dot{x}^2 = \dot{x}_\mu \dot{x}^\mu = \frac{ds_\mu}{d\tau}\frac{ds^\mu}{\tau} = \left(\frac{ds}{d\tau}\right)^2 = (1+2\Phi(r))\left(\frac{dt}{d\tau}\right)^2 - (1-2\Phi(r))\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right)$$

$$= (1+2\Phi(r))\,\dot{t}^2 - (1-2\Phi(r))(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$$

$$S = -mc\int d\tau \left[(1+2\Phi(r))\,\dot{t}^2 - (1-2\Phi(r))(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)\right]^{\frac{1}{2}} \quad ✓$$

b) You can now read off a Lagrangian L. Why could it also be given as

$$L = (1+2\Phi(r))\,\dot{t}^2 - (1-2\Phi(r))(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)\ ?$$

We read off $L = -mc\left[(1+2\Phi(r))\,\dot{t}^2 - (1-2\Phi(r))(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)\right]^{\frac{1}{2}}$. ✓

In exercise 2 of sheet 1, we showed that the e.o.m.s don't change under transformations of the type

$$\mathcal{L}(q,\dot{q},t) \longrightarrow \mathcal{L}'(q,\dot{q},t) = \alpha\,\mathcal{L}(q,\dot{q},t) + \frac{d}{dt}f(q,t).$$

In our case, $\alpha_1 = -mc$ can therefore be dropped. A square also has no effect on the e.o.m.s:

$$0 \overset{!}{=} \delta S = \delta\int d\tau\, L = \int d\tau\, \delta L \iff 0 \overset{!}{=} \delta S = \delta\int d\tau \sqrt{L} = \int d\tau\, \underbrace{\frac{\delta L}{2\sqrt{L}}}_{\alpha_2} \quad ✓$$

Applying a variation to S, the square root gives just another scaling factor α_2.

→ if L is constant, which is given for the affine parameter τ

c) Compute the equations of motion from the Euler-Lagrange equations. (Hint: Only keep terms that are linear in Φ or quadratic in $\dot{q}_i$.)

r: $$\frac{d}{d\tau}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = -\frac{d}{d\tau}\left[2\left(1-2\Phi(r)\right)\dot r\right] - 2\frac{\partial \Phi(r)}{\partial r}\dot t^2 - 2\frac{\partial\Phi(r)}{\partial r}\left(\dot r^2 + r\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2\right)$$

$$+2\left(1-2\Phi(r)\right) r\dot\theta^2 + 2\left(1-2\Phi(r)\right) r\sin^2\theta\,\dot\phi^2 = 0$$

$$\Longleftrightarrow \ddot r - \frac{\Phi(r)\dot t^2}{(1-2\Phi(r))r} - \frac{\Phi(r)}{(1-2\Phi(r))r}\left(\dot r^2 + r\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2\right) - r\dot\theta^2 - r\sin^2\theta\,\dot\phi^2 - \frac{2\Phi(r)}{(1-2\Phi(r))r}\dot r^2 = 0$$

$$\Longleftrightarrow \ddot r - r\dot\theta^2 - r\sin^2\theta\,\dot\phi^2 - \frac{\Phi(r)}{(1-2\Phi(r))r}\left(\dot t^2 - \dot r^2 + r\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2\right) = 0 \quad ✓$$

θ: $$\frac{d}{d\tau}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = -2\frac{d}{d\tau}\left(1-2\Phi(r)\right)r^2\dot\theta + 2\left(1-\Phi(r)\right)r^2\sin\theta\cos\theta\,\dot\phi^2 = 0$$

$$\Longleftrightarrow 4\frac{\partial\Phi(r)}{\partial r}\dot r r^2\dot\theta - 4\left(1-2\Phi(r)\right)r\dot r\dot\theta - 2\left(1-2\Phi(r)\right)r^2\ddot\theta + 2\left(1-2\Phi(r)\right)r^2\sin\theta\cos\theta\,\dot\phi^2 = 0$$

$$\Longleftrightarrow \ddot\theta + \frac{2\Phi(r)\dot r\dot\theta}{(1-2\Phi(r))r} + 2\frac{\dot r}{r}\dot\theta - \sin\theta\cos\theta\,\dot\phi^2 = \ddot\theta + 2\frac{\dot r}{r}\dot\theta\,\frac{1-\Phi(r)}{1-2\Phi(r)} - \sin\theta\cos\theta\,\dot\phi^2 = 0$$ ✓

ϕ: $$\frac{d}{d\tau}\frac{\partial L}{\partial\dot\phi} - \frac{\partial L}{\partial\phi} = \frac{d}{d\tau}\left(-2\left(1-2\Phi(r)\right)r^2\sin^2\theta\,\dot\phi\right) = 4\frac{\partial\Phi(r)}{\partial r}\dot r r^2\sin^2\theta\,\dot\phi - 4\left(1-2\Phi(r)\right)r\dot r\sin^2\theta\,\dot\phi$$

$$-4\left(1-2\Phi(r)\right)r^2\sin\theta\cos\theta\,\dot\theta\dot\phi - 2\left(1-2\Phi(r)\right)r^2\sin^2\theta\,\ddot\phi = 0$$

$$\Longleftrightarrow \ddot\phi + \frac{2\Phi(r)}{(1-2\Phi(r))r}\dot r\dot\phi + 2\frac{\dot r}{r}\dot\phi + 2\cot\theta\,\dot\theta\dot\phi = \ddot\phi + 2\frac{\dot r}{r}\dot\phi\,\frac{1-\Phi(r)}{1-2\Phi(r)} + 2\cot\theta\,\dot\theta\dot\phi = 0$$ ✓

In the above simplifications, we used that $\frac{\partial\Phi(r)}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{GM}{c^2 r}\right) = \frac{GM}{c^2r^2} = -\frac{\Phi(r)}{r}$.

What about $x^0 = t$?

d) By identifying your equations of motion with the geodesic equation (1), find all non-trivial Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$.

$x^1 = r$: $\Gamma^1{}_{00} = -\Gamma^1{}_{11} = -\frac{1}{r}\frac{\Phi(r)}{1+2\Phi(r)}$ ✓, $\Gamma^1{}_{22} = -r\left(1+\frac{\Phi(r)}{1+2\Phi(r)}\right)$ ✓, $\Gamma^1{}_{33} = -r\sin^2\theta\left(1+\frac{\Phi(r)}{1+2\Phi(r)}\right)$ ✓

$x^2 = \theta$: $\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r}\frac{1-\Phi(r)}{1+2\Phi(r)}$, $\Gamma^2{}_{33} = -\sin\theta\cos\theta$ ✓

$x^3 = \phi$: $\Gamma^3{}_{12} = \Gamma^3{}_{12} = \frac{1}{r}\frac{1-\Phi(r)}{1-2\Phi(r)}$, $\Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta$ ✓, all other Christoffel symbols zero

Problem 4 (Weak field II: Let's lens like its 1999) [10 points]

We've seen that in a weak-field limit, the metric tensor can be written as

$$g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -\left(1+\frac{2\Phi}{c^2}\right) & 0 & 0 \\ 0 & 0 & -\left(1+\frac{2\Phi}{c^2}\right) & 0 \\ 0 & 0 & 0 & -\left(1+\frac{2\Phi}{c^2}\right) \end{pmatrix}, \qquad (2)$$

where Φ is the Newtonian gravitational potential. From this, and from Fermat's principle, we're going to derive the correct gravitational lensing deflection angle after having seen the Newtonian case on practice sheet 1.

a) What are the assumptions that go into finding this limit?

The metric in eq. (2) can be derived in the limit of slowly moving particles ($v \ll c$) and a weak (gravitational) field by employing the 'infinitesimal' Ansatz ✓

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \text{where } |h_{\mu\nu}| \ll 1. \quad ✓$$

Write down the line element $ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$ in cartesian coordinates (ct, x, y, z).

$$ds^2 = (1+2\Phi/c^2)c^2dt^2 - (1-2\Phi/c^2)(dx^2+dy^2+dz^2) = (1+2\Phi/c^2)c^2dt^2 - (1-2\Phi/c^2)\,d\vec{x}^2 \quad ✓$$

b) What special sort of geodesics do light rays follow, i.e. what value does ds have for light?

Light rays follow null geodesics, i.e. $ds = \sqrt{ds^2} = 0$. ✓

c) From part b), you can infer the effective velocity of the light ray in the gravitational field, $c' = \frac{d\vec{x}}{dt}$. Approximate it, using that for $a \ll 1$,

$$\sqrt{1+a}\,/\,\sqrt{1-a} \approx (1+a).$$

The refractive index, $n = \frac{c}{c'}$, can now be written down. Use $(1+a)^{-1} \approx 1-a$.

$$0 \overset{!}{=} ds^2 = \Big[(1+2\Phi/c^2)\,c^2 - (1-2\Phi/c^2)\underbrace{\frac{d\vec{x}^2}{dt^2}}_{c'^2}\Big]\,dt^2 \implies c' = \sqrt{\frac{1+2\Phi/c^2}{1-2\Phi/c^2}}\,c \approx (1+2\Phi/c^2)\,c \quad ✓$$

$$n = \frac{c}{c'} = \frac{c}{(1+2\Phi/c^2)\,c} \approx 1-2\Phi/c^2 \quad ✓$$

d) Fermat's principle states that light takes an optically extremal path, i.e.

$$\delta \int dl\, n(\vec{x}(l)) = 0 \quad ✓$$

We can write $dl = \left|\frac{d\vec{x}}{d\lambda}\right| d\lambda = |\dot{\vec{x}}|\, d\lambda$, where we defined $\dot{\vec{x}} := d\vec{x}/d\lambda$. Put this into Fermat's principle, which should look like the variation of an action to you now,

$$\delta \int dt\, L(\vec{x}, \dot{\vec{x}}, t) = 0 \quad ✓$$

What is the 'Lagrangian' $L(\vec{x}, \dot{\vec{x}}, \lambda)$ in our case

$$\delta \int dl\, n(\vec{x}(l)) = \delta \int d\lambda\, |\dot{\vec{x}}|\, n(\vec{x}(\lambda)) = 0 \quad \Longrightarrow \quad L(\vec{x}, \dot{\vec{x}}, \lambda) = |\dot{\vec{x}}|\, n = |\dot{\vec{x}}(\lambda)|\, n(\vec{x}(\lambda)) \quad ✓$$

e) Show that applying the Euler-Lagrange eqs.

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\partial L}{\partial \vec{x}},$$

yields something of the form

$$\frac{d}{d\lambda}\left(n \vec{e}_{\dot{x}}\right) = (\vec{\nabla} n)\, |\dot{\vec{x}}|, \qquad (3)$$

where $\vec{\nabla} n \equiv \frac{\partial}{\partial \vec{x}} n$ and $\vec{e}_{\dot{x}}$ is the unit vector pointing in $\dot{\vec{x}}$-direction. We can set $|\dot{\vec{x}}| = 1$ for simplicity because the scale of λ is arbitrary.

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} = \frac{d}{d\lambda}\underbrace{\frac{\partial}{\partial \dot{\vec{x}}}}_{\vec{e}_{\dot{x}}\partial/\partial|\dot{\vec{x}}|}\left(|\dot{\vec{x}}(\lambda)|\, n(\vec{x}(\lambda))\right) = \frac{d}{d\lambda}\left(\vec{e}_{\dot{x}}\, n(x(\lambda))\right) ✓, \quad \frac{\partial L}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}}\left(|\dot{\vec{x}}(\lambda)|\, n(\vec{x}(\lambda))\right) = (\vec{\nabla} n)|\dot{\vec{x}}| ✓$$

$$\Longrightarrow \quad \frac{d}{d\lambda}\left(n \vec{e}_{\dot{x}}\right) = (\vec{\nabla} n)|\dot{\vec{x}}| \quad ✓$$

f) Simplify eq. (3) to the form (Hint: $\frac{d}{d\lambda} n = \frac{\partial n}{\partial \vec{x}} \frac{d\vec{x}}{d\lambda}$. Also use that $|\dot{\vec{x}}| = 1$, thus $\dot{\vec{x}} = \vec{e}_{\dot{x}}$)

$$n \dot{\vec{e}}_{\dot{x}} = \vec{\nabla} n - \vec{e}_{\dot{x}}(\vec{\nabla} n \cdot \vec{e}_{\dot{x}}).$$

$$\frac{d}{d\lambda}\left(n \vec{e}_{\dot{x}}\right) = \frac{dn}{d\lambda}\vec{e}_{\dot{x}} + n\frac{d\vec{e}_{\dot{x}}}{d\lambda} = \left(\frac{\partial n}{\partial \vec{x}} \cdot \frac{d\vec{x}}{d\lambda}\right)\vec{e}_{\dot{x}} + n\dot{\vec{e}}_{\dot{x}} = (\vec{\nabla} n \cdot \dot{\vec{x}})\,\vec{e}_{\dot{x}} + n\dot{\vec{e}}_{\dot{x}} = \vec{e}_{\dot{x}}(\vec{\nabla} n \cdot \vec{e}_{\dot{x}}) + n\dot{\vec{e}}_{\dot{x}} \quad ✓$$

Subtracting the first product-rule term to the left of eq. (3), we get: $n\dot{\vec{e}}_{\dot{x}} = \vec{\nabla} n - \vec{e}_{\dot{x}}(\vec{\nabla} n \cdot \vec{e}_{\dot{x}})$ ✓

g) Since the right-hand side is subtracting the light-ray direction of $\vec{\nabla} n$ from $\vec{\nabla} n$, the only remaining part is the one perpendicular to the ray, $\vec{\nabla}_\perp n$, such that

$$\dot{\vec{e}}_{\dot{x}} = \frac{1}{n}\vec{\nabla}_\perp n = \vec{\nabla}_\perp \ln n .$$

Plug in n from part c) and use $\ln(1-a) \approx a$. What do you get for $\dot{\vec{e}}_{\dot{x}}$?

$$\dot{\vec{e}}_{\dot{x}} = \vec{\nabla}_\perp \ln n = \vec{\nabla}_\perp \ln\left(1-2\Phi/c^2\right) \approx \vec{\nabla}_\perp\left(-2\Phi/c^2\right) = -\frac{2}{c^2}\vec{\nabla}_\perp \Phi \quad ✓$$

h) The deflection angle is given by $\hat{\alpha} = \int d\lambda \left(-\dot{\vec{e}}_{\dot{x}}\right)$. What do you get if you use your knowledge of $\dot{\vec{e}}_{\dot{x}}$ from part g)?

$$\hat{\alpha} = \int d\lambda\left(-\dot{\vec{e}}_{\dot{x}}\right) = \int d\lambda \frac{2}{c^2}\vec{\nabla}_\perp \Phi = \frac{2}{c^2}\int d\lambda \vec{\nabla}_\perp \Phi \quad ✓$$

i) Your resulting integral can be cumbersome, since we would have to continuously integrate over the light path. However, since we assume the deflection angle to be small, we can use the Born approximation from scattering theory, introducing the impact parameter ξ, and integrating over the unperturbed path

$$\hat{\alpha}(\xi) = \frac{2}{c^2}\int_{-\infty}^{\infty} d\lambda \; \vec{\nabla}_\perp \Phi .$$

For simplicity, we assume that ξ lies in the x-y-plane. Then the distance to a point-like mass at the origin is $r^2 = \xi^2 + \lambda^2$ (because λ is evaluated at the unperturbed path). Taking $\Phi = -\frac{GM}{r}$, we get $\vec{\nabla}_\perp \Phi = \frac{GM}{r^3}\xi$ and thus for the deflection angle

$$\hat{\alpha}(\xi) = \frac{2GM}{c^2}\xi \int_{-\infty}^{\infty} d\lambda \frac{1}{r^3} .$$

Evaluate the integral. $\hat{\alpha}(\xi)$ is then the fully relativistic gravitational lensing deflection angle.

First we reproduce $\vec{\nabla}_\perp \Phi$:

$$\vec{\nabla}_\perp \Phi = \vec{\nabla}_\perp\left(-\frac{GM}{r}\right) = \frac{GM}{r^2}\vec{\nabla}_\perp r = \frac{GM}{r^2}\underbrace{\vec{\nabla}_\perp}_{\hat{=}\frac{d}{d\xi}}\sqrt{\xi^2+\lambda^2} = \frac{GM}{r^2}\frac{2\xi}{2\underbrace{\sqrt{\xi^2+\lambda^2}}_{r}} = \frac{GM}{r^3}\xi \quad ✓$$

We now move on to evaluate the integral using substitution,

$$I := \int_{-\infty}^{\infty} d\lambda \frac{1}{r^3} = \int_{-\infty}^{\infty} d\lambda \frac{1}{(\xi^2+\lambda^2)^{\frac{3}{2}}}$$

We know all occuring values to be positive so that we may substitute,

$$\lambda = \xi_1 \tan(u), \quad d\lambda = \frac{\xi_1}{\cos^2 u} du, \quad u(\lambda=\infty) = \arctan\left(\frac{\infty}{\xi_1}\right) = \frac{\pi}{2}, \quad u(\lambda=-\infty) = \arctan\left(\frac{-\infty}{\xi_1}\right) = -\frac{\pi}{2},$$ ✓

giving the integrand $(\xi^2+\lambda^2)^{-\frac{3}{2}} = (\xi_1^2 + \xi_1^2 \tan^2 u)^{-\frac{3}{2}} = \frac{1}{\xi_1^3}(1+\tan^2 u)^{-\frac{3}{2}} = \frac{1}{\xi_1^3}\cos^3 u$. Therefore,

$$I = \int_{-\infty}^{\infty} d\lambda \frac{1}{(\xi_1^2+\lambda^2)^{\frac{3}{2}}} = \int_{-\pi/2}^{\pi/2} du \frac{\xi_1}{\cos^2 u} \cdot \frac{1}{\xi_1^3} \cos^3 u = \frac{1}{\xi_1^2} \int_{-\pi/2}^{\pi/2} \cos u \, du = \frac{1}{\xi_1^2} \sin u \Big|_{-\pi/2}^{\pi/2} = \frac{2}{\xi_1^2}$$ ✓

Consequently, the relativistic lensing deflection angle $\hat{\alpha}(\xi_1)$ is given by

$$\hat{\alpha}(\xi_1) = \frac{2GM}{c^2} \xi_1 I = \frac{2GM}{c^2} \xi \frac{2}{\xi_1^2} = \frac{4GM}{c^2 \xi_1}.$$ ✓

Problem 5 (Extra question: Sym-Metric) [5 points]

Why does the metric tensor $g_{\mu\nu}$ necessarily have to be symmetric? What implications would asymmetry have on the equations of motion?

As soon as one has a distance of concept, one can define an inner product and from this the metric follows and inherits its symmetry. Since $x \cdot y = y \cdot x$, we can require

✓

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu \overset{!}{=} g_{\nu\mu} y^\nu x^\mu = y \cdot x \implies g_{\mu\nu} = g_{\nu\mu}$$