

Group Theory - Problem Sheet 3Exercise 1 (Representations of S_n)

- a) Use the hook rule to determine the dimension of the irreducible representation associated to the partition

$$\lambda = (n-k, \underbrace{1, \dots, 1}_{k \text{ times}}, 0, \dots, 0)$$

The hook length of a box in a Young tableau is the number of boxes beneath or to the right of the box in the diagram, counting the box itself.

Let ρ^λ be the irreducible repr. of S_n associated with the partition λ , then the hook rule states the dimension of ρ^λ to be $n!$ over the hook lengths in the Young diagram of shape λ , i.e. in this instance

$$\dim(\rho^\lambda) = \frac{n!}{(n-k+k)} = (n-1)! \quad \text{Young diagram: } \left. \begin{array}{c} \overbrace{\quad \dots \quad}^{n-k \text{ boxes}} \\ \vdots \\ \end{array} \right\} k \text{ boxes}$$

- b) Construct the character table of S_n by using the Frobenius formula.

According to the Frobenius formula the characters of the irreducible repr. of S_n associated with the partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n is given by

$$\chi_\lambda(C_i) = \left[\frac{\Lambda(x)}{\prod_{j=1}^k P_j(x)^{r_j}} \right]_{(\lambda_1, \dots, \lambda_k)}$$

where P_j is the power sum $P_j(x) = x_1^j + \dots + x_n^j$, k being the number of rows in the Young tableau of λ , and $\Lambda(x)$ is the discriminant

$$\Lambda(x) = \prod_{1 \leq i < j \leq k} (x_i - x_j) = \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & \dots & x_k^{k-1} \end{pmatrix}$$

of the set of k independent variables $\{x_1, \dots, x_k\}$. Furthermore, $i = (i_1, \dots, i_n)$ is an n -tuple of non-negative integers s.t. $\sum_r r_i = n$ so that C_i may denote the conjugacy class consisting of elements

with cycle type $(i_1, \dots, i_n)$, and $[f(x)]_{l_1, \dots, l_n}$ is the coefficient of $x_1^{l_1} \dots x_n^{l_n}$ in the power series $f(x) = f(x_1, \dots, x_n)$, where $L = (l_1, \dots, l_n)$ is a k -tuple of non-negative integers.

Applying the Frobenius formula yields the following character table

S_4	$C_1 = [1]$	$C_2 = [12]$	$C_3 = [123]$	$C_4 = [1234]$	$C_5 = [12434]$
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ	3	0	0	-1	1
$\chi_{\text{sgn}} \otimes \chi$	3	-1	0	1	-1
χ_{other}	2	0	-1	0	2

Exercise 2 (Lie groups and Lie algebras)

f) How are the notions of homomorphisms of Lie groups and Lie algebras related?

Let $\phi: G \rightarrow H$ be a Lie group homomorphism. With ϕ_* we denote its derivative at the identity of G . If we pick $\mathfrak{g}, \mathfrak{h}$, the Lie algebras of G and H , as the tangent spaces at their respective identities, then ϕ_* is a map between these two Lie algebras,

$$\phi_*: \mathfrak{g} \rightarrow \mathfrak{h},$$

which fulfills all the requirements of a Lie group homomorphism, i.e. it is a linear map that preserves the Lie bracket.

Exercise 2 (Lie groups and Lie algebras)

a) What is a Lie group?

A Lie group is a group G which is also a differentiable (also called smooth) manifold s.t. the group operations are compatible with the smooth group structure, i.e. s.t.

$$\left. \begin{array}{l} m: G \times G \rightarrow G \quad (\text{multiplication}) \\ (\cdot)^{-1}: G \rightarrow G \quad (\text{inversion}) \end{array} \right\} \text{ are smooth maps}$$

b) What is a Lie algebra?

A Lie algebra is a vector space V with a non-associative, bilinear, skew-symmetric map $[\cdot, \cdot]: V \times V \rightarrow V$ (called the Lie bracket) which satisfies the Jacobi-identity.

c) What are left-invariant vector fields? Describe them for matrix groups.

A vector field X_v on a Lie group G is left-invariant if

$$(dL_g)(X_v(h)) = X_v(L_g(h)) = X_v(gh), \quad \forall g, h \in G$$

where dL_g is the derivative of the smooth map L_g , and L_g denotes the group action of the Lie group G defined by left multiplication, i.e. $L_g: G \rightarrow G$, $L_g(h) = gh \quad \forall g, h \in G$ ($L_e = \text{id}_G$).

d) How does one obtain the structure of a Lie algebra on a tangent space of a Lie group at the identity?

By finding the vector space of all left-invariant vector fields.

Reasoning: Since the vector space of all left-invariant vector fields is isomorphic (as a vector space) to $T_e G$, the tangent of a Lie group G at the identity e , and it in turn is equipped with the structure of a Lie algebra, defined by

$$[X, Y] = X_{D,Y} = [X_Y, X_X] \quad \forall X, Y \in T_e G,$$

this yields the desired information.

e) Show that the Lie bracket on the tangent space of the identity of matrix groups satisfies the Jacobi identity.

Every matrix (Lie) group is a subgroup of the general linear group $GL(n, \mathbb{K})$

$$GL(n, \mathbb{K}) = \{A \in \text{Mat}(n, n, \mathbb{K}) \mid \det(A) \neq 0\},$$

i.e. the group of all invertible $n \times n$ -matrices over $\mathbb{K}$.

Since the (standard) Lie bracket for Lie algebras of the matrix groups is simply the matrix commutator given by $[A, B] = AB - BA$ for $A, B \in GL(n, \mathbb{K})$, we calculate

$$\begin{aligned} [A, [B, C]] + [B, [C, A]] + [C, [A, B]] &= ABC - BCA - \underline{ACB} - \underline{CBA} \\ &+ \underline{BCA} - \underline{CAB} - \underline{BAC} + \underline{ACB} + \underline{CAB} - \underline{ABC} - \underline{CBA} + \underline{BAC} = 0, \end{aligned}$$

which holds for all $A, B, C \in GL(n, \mathbb{K})$ and therefore for all possible elements of matrix groups.

Exercise 3 (SU(2))

Show that $SU(2) = \{A \in \text{Mat}(2, 2, \mathbb{C}) \mid A^* A = 1, \det A = 1\}$ can be parametrized in the following way

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

(Hence, $SU(2)$ is homeomorphic to the three sphere S^3 , and in particular simply connected.)

We write an element $A \in SU(2)$ as $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}$. Then

$$A^* A = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & \bar{a}c + \bar{b}d \\ a\bar{c} + b\bar{d} & |c|^2 + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The above property of A as an element $SU(2)$ yields the following three constraints on $a, b, c, d \in \mathbb{C}$,

$$\text{i) } |a|^2 + |b|^2 = 1 \quad \text{ii) } |c|^2 + |d|^2 = 1 \quad \text{iii) } a\bar{c} + b\bar{d} = 0$$

The condition $\det(A) = 1$ further supplies

$$\text{iv) } ad - bc = 1$$

From iii), $A = -\frac{b\bar{d}}{c}$ follows. Inserting into i), we obtain

$$\frac{|b|^2 |d|^2}{|c|^2} + |b|^2 = 1 \Rightarrow \frac{|c|^2}{|b|^2} = |c|^2 + |d|^2 = 1,$$

i.e. $|b| = |c|$. Similarly, we find $|a| = |d|$ by subtracting i) and ii)

$$|a|^2 + \underbrace{|b|^2}_{=|c|^2} - |c|^2 - |d|^2 = |a|^2 - |d|^2 = 0.$$

Remark: We assumed above, that $b \neq 0$. If $b = 0$, it becomes very

simple. By iv), we know $ad = 1$ and by iii), $a\bar{c} = 0$. The

latter implies $c = 0$, since $a \neq 0$, and A becomes diagonal

with $a = \bar{d}$ because $ad = 1$.

Now, returning to the case $b \neq 0$, we discover that our findings are captured by the ansatz

$$a = e^{i\alpha} \cos \theta, \quad b = e^{i\beta} \sin \theta, \quad c = -e^{i\gamma} \sin \theta, \quad d = e^{i\delta} \cos \theta,$$

where $\alpha, \beta, \gamma, \delta, \theta \in \mathbb{R}$. Applying iv) to this ansatz,

$$ad - bc = e^{i(\alpha+\delta)} \cos^2 \theta + e^{i(\beta+\gamma)} \sin^2 \theta = 1 \iff \alpha = -\delta, \beta = -\gamma,$$

we finally arrive at

$$a = e^{i\alpha} \cos \theta = e^{-i\delta} \cos \theta = \bar{d}, \quad b = e^{i\beta} \sin \theta = e^{-i\gamma} \sin \theta = -\bar{c}.$$

Thus, $SU(2)$ may be parametrized by

$$SU(2) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & -\bar{c} \\ b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

Exercise 4 (SU(2) versus SO(3))

a) Determine the Lie algebras of $SU(2)$ and $SO(3)$ and show that they are isomorphic.

The $SU(2)$ Lie algebra is

$$\mathfrak{su}(2) = \{A \in \text{Mat}(2, 2, \mathbb{C}) \mid \text{tr}(A) = 0, A^* = -A\}$$

$$= \left\{ \begin{pmatrix} ia & -\bar{z} \\ z & -ia \end{pmatrix} \mid a \in \mathbb{R}, z \in \mathbb{C} \right\},$$

which amounts to the vector space of all traceless, antihermitian 2×2 -matrices over $\mathbb{C}$. It is $n^2 - 1 = 3$ -dimensional and generated by the following matrices

$$u_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The Lie algebra of $SO(3)$,

$$so(3) = \{A \in Mat(3,3,\mathbb{R}) \mid A^T = -A\} = \left\{ \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix} \mid w_x, w_y, w_z \in \mathbb{R} \right\}$$

is the vector space of all antisymmetric 3×3 -matrices over $\mathbb{R}$.

$so(3)$ is $\frac{n(n-1)}{2} = 3$ -dimensional with a generating basis formed by the following three matrices

$$o_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad o_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad o_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Both assertions concerning the structure of the Lie algebras $su(2)$ and $so(3)$ can be seen by selecting and differentiating smooth curves from their corresponding Lie groups:

- Let $U(t) \in SU(2)$ be a smooth curve with $U(0) = 1_2$. Then $U(t)$ satisfies

$$U(t) U(t)^T = 1_2, \quad \det(U(t)) = 1.$$

Differentiation yields

$$\frac{dU^T(t)}{dt} U(t) + U^T(t) \frac{dU(t)}{dt} = 0, \quad \text{tr}\left(U^T(t) \frac{dU(t)}{dt}\right) = 0,$$

which, for $t=0$, and writing $\left. \frac{dU(t)}{dt} \right|_{t=0}$ as $u \in su(2)$, demonstrates the above claims.

$$u^T + u = 0, \quad \text{tr}(u) = 0.$$

- Let $O(t) \in SO(3)$ be a smooth curve with $O(0) = 1_3$. Then $O(t)$ satisfies the defining relation of $O(3)$ -elements $O^T(t) O(t) = O$. Thus

$$\frac{dO^T(t)}{dt} O(t) + O^T(t) \frac{dO(t)}{dt} = 0$$

and so for $t=0$ and $o := \left. \frac{dO(t)}{dt} \right|_{t=0}$, $o^T + o = 0$.

Now on to the isomorphism between $su(2)$ and $so(3)$. We can show this relation using the group homomorphism $\Phi: SU(2) \rightarrow SO(3)$ to construct an isomorphism $\phi: su(2) \rightarrow so(3)$.

We start by expressing Φ using abstract index notation. Let $f(\vec{x})$ be a function of position $\vec{x}$, $R \in SO(3)$ a rotation matrix and U a unitary transformation, the image of $\Phi(R)$ in $SU(2)$. Then

$$f(R\vec{x}) = f(\Phi(U)\vec{x}) = U f(\vec{x}) U^\dagger,$$

$$R_j x^i \sigma_i = U x^i \sigma_i U^\dagger.$$

Differentiating at $t=0$ and setting $\frac{dR}{dt}|_{t=0} = \underline{X} \in so(3)$, $U(0) = 1$, and $\frac{dU}{dt}|_{t=0} = \underline{A} = a^i \sigma_i \in su(2)$, we obtain

$$\underline{X}_j x^i \sigma_i = \underline{A} x^i \sigma_i + x^i \sigma_i \underline{A}^\dagger$$

$$x^i \underline{X}_j \sigma_i = x^i (\underline{A} \sigma_i - \sigma_i \underline{A}) = x^i [\underline{A}, \sigma_i] = x^i [a^k \sigma_k, \sigma_i]$$

$$= -\frac{i}{2} x^i a^k [\sigma_i, \sigma_k] = -\frac{i}{2} x^i a^k (2i \epsilon_{ijk} \sigma_k)$$

$$= x^i a^k \epsilon_{ijk} \sigma_k = x^i (-a^k \epsilon_{kij}) \sigma_i$$

$$=: x^i \phi(\underline{A})_j^i \sigma_i$$

So the isomorphism between $su(2)$ and $so(3)$ is given by

$$\phi(\underline{A})_j^i = \epsilon_{jki} a^k : su(2) \rightarrow so(3).$$

With this knowledge, we can say that because Lie algebras are locally identical to their corresponding Lie groups, $SU(2)$ and $SO(3)$ are locally isomorphic.