String Theory

Solution to Assignment 3

Janosh Riebesell

October 26th, 2015 (due November 4th, 2015)

Lecturer: Timo Weigand

1 Mixed Dirichlet-Neumann boundary conditions

The derivation of the mode expansion for the open string with Neumann-Neumann NN boundary conditions is easiest performed with the help of the so-called doubling trick: Starting from the open string field $X_\mu(\tau,\sigma)$ with $0 \le \sigma \le l$, one defines the auxiliary field $\hat{X}_\mu(\tau,\sigma)$ for $-l \le \sigma \le l$ via

$$\hat{X}_\mu(\tau,\sigma) = \begin{cases} X_\mu(\tau,\sigma) & \text{for } 0 \le \sigma \le l, \\ X_\mu(\tau,-\sigma) & \text{for } -l \le \sigma \le 0. \end{cases} \tag{1}$$

a) Show that requiring that $\hat{X}_\mu(\tau,\sigma)$ can be extended to $\sigma \in \mathbb{R}$ as a smooth periodic function with period $2l$ is equivalent to requiring Neumann-Neumann boundary conditions for $X_\mu(\tau,\sigma)$ at $\sigma \in \{0,l\}$.

b) Use this to deduce the oscillator expansion for the Neumann-Neumann open string. Start with a solution for $\partial_\pm \hat{X}_\mu(\tau,\sigma)$ and integrate this to a solution for $X_\mu(\tau,\sigma)$ subject to the correct boundary conditions.

c) Repeat this procedure for Dirichlet-Dirichlet DD boundary conditions by first finding a suitable definition of $\hat{X}_\mu(\tau,\sigma)$ that incorporates the DD boundary conditions for $X_\mu(\tau,\sigma)$ and then integrating the solution for $\partial_\pm \hat{X}_\mu(\tau,\sigma)$.

d) Find the open string expansion for DN boundary conditions (i.e. Dirichlet boundary conditions at $\sigma = 0$ and Neumann boundary conditions at $\sigma = l$).

e) For each of the three types of boundary conditions - NN, DD, DN - give the centre-of-mass position and the total momentum of the string and discuss the result.

a) Since $\hat{X}_\mu(\tau,\sigma)$ is symmetric around $\sigma = 0$, by the chain rule, its derivative has to be antisymmetric,

$$\hat{X}_\mu(\tau,\sigma) = \hat{X}_\mu(\tau,-\sigma) \quad \Rightarrow \quad \partial_\sigma \hat{X}_\mu(\tau,\sigma) = -\partial_\sigma \hat{X}_\mu(\tau,-\sigma). \tag{2}$$

This immediately implements the Neumann boundary condition for the real string field $X_\mu(\tau,\sigma)$ at $\sigma = 0$,

$$\partial_\sigma X_\mu(\tau,0) = \partial_\sigma \hat{X}_\mu(\tau,0) \overset{!}{=} -\partial_\sigma \hat{X}_\mu(\tau,0) = -\partial_\sigma X_\mu(\tau,0) \quad \Rightarrow \quad \partial_\sigma X_\mu(\tau,0) = 0. \tag{3}$$

By further taking into account periodicity of the auxiliary field, i.e. $X_\mu(\tau,l) \overset{!}{=} X_\mu(\tau,-l)$, we get exactly the same condition at $\sigma = l$,

$$\partial_\sigma X_\mu(\tau,l) = \partial_\sigma \hat{X}_\mu(\tau,l) \overset{!}{=} -\partial_\sigma \hat{X}_\mu(\tau,-l) = -\partial_\sigma X_\mu(\tau,l) \quad \Rightarrow \quad \partial_\sigma X_\mu(\tau,l) = 0. \tag{4}$$

b) We prefer to work with the actual string field $X_\mu(\tau,\sigma)$ here as opposed to the auxiliary $\hat{X}_\mu(\tau,\sigma)$. We will have to take care to implement the boundary conditions by hand but this has the advantage of making it clearer what is going on.

To deduce the string field's mode expansion, we first need its equation of motion. It can be obtained from the Polyakov action, which in flat gauge ($h_{ab} = \eta_{ab}$) becomes

$$\begin{aligned} S_\mathrm{P} &= -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\,\sqrt{-h}\,h^{ab}\partial_a X_\mu \partial_b X^\mu \overset{h_{ab}=\eta_{ab}}{\longrightarrow} -\frac{T}{2}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\,\underbrace{\sqrt{-\eta}}_{1}\,\eta^{ab}\partial_a X_\mu \partial_b X^\mu \\ &= \frac{T}{2}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\,\Big[(\partial_\tau X)^2 - (\partial_\sigma X)^2\Big] = 2T\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\,\partial_+ X_\mu \partial_- X^\mu, \end{aligned} \tag{5}$$

where in the last step, we used lightcone coordinates $\xi^\pm = \tau \pm \sigma$ in which the partial derivatives read $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. By varying S_P w.r.t. $X^\mu(\tau,\sigma)$, we get the string field's e.o.m.,

$$\begin{aligned} \delta S_\mathrm{P} &= \frac{T}{2}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\,\Big[2\partial_\tau X_\mu\,\partial_\tau \delta X^\mu - 2\partial_\sigma X_\mu\,\partial_\sigma \delta X^\mu\Big] \\ &= -T\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\,\Big[\partial_\tau^2 X_\mu - \partial_\sigma^2 X_\mu\Big]\delta X^\mu \\ &\quad + T\int_0^l \mathrm{d}\sigma\,\partial_\tau X_\mu\,\delta X^\mu\Big|_{t_i}^{\tau=t_f} - T\int_{t_i}^{t_f}\mathrm{d}\tau\,\partial_\sigma X_\mu\,\delta X^\mu\Big|_0^{\sigma=l}. \end{aligned} \tag{6}$$

We drop the two boundary terms in the last line because we desperately want to avoid non-local terms in the string field's e.o.m. Actually, this step isn't as unfounded as we just made it sound. As usual, our variation underlies the constraint that the initial and final string state are held fixed, i.e. $\delta X^\mu(\tau = t_i, \sigma) = \delta X^\mu(\tau = t_f, \sigma) = 0$, so the τ-boundary takes care of itself in any case. At least for the *closed* string, the σ-boundary is trivial as well: There simply is none. Instead we have periodic the boundary condition $X^\mu(\tau, \sigma = l) \overset{!}{=} X^\mu(\tau, \sigma = 0)$. The part where it becomes interesting is the σ-boundary of the *open* string. Since now there is no periodicity in σ, we need $\partial_\sigma X_\mu\,\delta X^\mu = 0$ at $\sigma = l$ and $\sigma = 0$ separately. This amounts to requiring (for each spacetime dimension μ individually) either

$$\partial_\sigma X^\mu(\tau,\sigma) = 0 \qquad \text{at } \sigma = l \text{ and/or } 0 \qquad \text{(Neumann b.c.)}, \tag{7}$$

$$\text{or} \qquad \delta X^\mu(\tau,\sigma) = 0 \qquad \text{at } \sigma = l \text{ and/or } 0 \qquad \text{(Dirichlet b.c.)}. \tag{8}$$

If we do this, then by the usual arguments of $\delta X^\mu(\tau,\sigma)$ being an arbitrary variation but δS_P having to vanish for all of them, we obtain the as string field e.o.m. the nice and simple result of a free wave equation $(\partial_\tau^2 - \partial_\sigma^2)X^\mu(\tau,\sigma) = 0$. Expressed in lightcone coordinates, this becomes

$$\partial_+\partial_- X^\mu(\xi^+,\xi^-) = 0. \tag{9}$$

This is all well and good but was done mostly for completeness sake. Our task in this exercise is actually to derive the NN open string mode expansion. Let's start.

Since partial derivatives commute, we can draw two interesting conclusions from (9):

$$\partial_+\partial_- X^\mu(\xi^+,\xi^-) = 0 \quad \Rightarrow \quad \partial_- X^\mu(\cancel{\xi^+},\xi^-) = \partial_- X^\mu(\xi^-), \tag{10}$$

$$\partial_-\partial_+ X^\mu(\xi^+,\xi^-) = 0 \quad \Rightarrow \quad \partial_+ X^\mu(\xi^+,\cancel{\xi^-}) = \partial_+ X^\mu(\xi^+). \tag{11}$$

This means that $X^\mu(\xi^+,\xi^-)$ is the sum of a left- and a right-moving wave along the string, i.e.

$$X^\mu(\xi^+,\xi^-) = X^\mu_L(\xi^+) + X^\mu_R(\xi^-). \tag{12}$$

Equation (12) is the most general ansatz admitted by the partial differential equation (9). We specialize it to describe the NN open string by implementing the Neumann boundary conditions at both ends. Firstly, at $\sigma = 0$, we have

$$\partial_\sigma X^\mu_\mathrm{NN}(\tau,\sigma)\big|_{\sigma=0} = \frac{\partial X^\mu_L(\xi^+)}{\partial\xi^+}\frac{\partial\xi^+}{\partial\sigma}\Big|_{\sigma=0} + \frac{\partial X^\mu_L(\xi^-)}{\partial\xi^-}\frac{\partial\xi^-}{\partial\sigma}\Big|_{\sigma=0} = \partial_+ X^\mu_L(\tau) - \partial_- X^\mu_R(\tau) = 0, \tag{13}$$

where $\partial_- X_R^\mu$ picked up a minus sign due to the chain rule. Since their σ-derivatives are equal, the functions X_L^μ and X_R^μ have to be identical up to an additive constant c^μ.[1] By replacing $X_R^\mu = X_L^\mu + c^\mu$ and redefining $X_L^\mu \to X_L^\mu + \frac{c^\mu}{2}$, eq. (12) becomes

$$X_{\mathrm{NN}}^\mu(\xi^+, \xi^-) = X_L^\mu(\xi^+) + X_L^\mu(\xi^-). \tag{14}$$

Essentially, this equation states that left-moving waves are reflected into right-moving ones at the left boundary and vice versa, so it suffices to define one type of mover and we kept X_L^μ. For the boundary at $\sigma = l$, we get

$$\partial_\sigma X_{\mathrm{NN}}^\mu(\tau, \sigma)\big|_{\sigma=l} = \partial_+ X_L^\mu(\tau + l) - \partial_- X_L^\mu(\tau - l) = 0. \tag{15}$$

Since this equation must hold for all τ, we learn that $\partial_\pm X_L^\mu$ is periodic in its *single* argument with period $2l$. This intermediate result is particularly important as it means that $\partial_\pm X_L^\mu$ fulfills the *Dirichlet conditions* which determine whether or not a function $f(x)$ can be expanded into a Fourier series. The Dirichlet conditions are:

1. $f(x)$ must be periodic.
2. $f(x)$ must be single-valued and continuous, except at a finite number of finite discontinuities.
3. $f(x)$ must have only a finite number of maxima and minima within a period.
4. The integral over one period of $|f(x)|$ must converge.

Periodicity was the only missing piece out of the above four. We are therefore entitled to make the following ansatz for $\partial_\pm X_L^\mu(\xi^\pm)$,

$$\partial_\pm X_L^\mu(\xi^\pm) = \frac{\pi}{l}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-\frac{2\pi i}{2l} n\xi^\pm}, \tag{16}$$

where the α_n^μ are the string field's modes, $2l$ is one period, and the awkward prefactor $\frac{\pi}{l}\sqrt{\alpha'/2}$ as well as the sign in the exponent was introduced to follow the lecture note's convention where the positive frequency modes correspond to $n < 0$. Note that we were not required to introduce two sets of Fourier coefficients, say $(\alpha_n^\pm)^\mu$ because we identified for the open NN string X_L^μ and X_R^μ as one and the same function right from the start. It is trivial to see that eq. (16) fulfills the original e.o.m. $\partial_\mp \partial_\pm X_L^\mu(\xi^\pm) = 0$.

The next step is to integrate eq. (16) w.r.t. $\xi^\pm$,

$$\begin{aligned} X_L^\mu(\xi^\pm) &= \int \mathrm{d}\xi^\pm\, \partial_\pm X_L^\mu(\xi^\pm) = \frac{\pi}{l}\sqrt{\frac{\alpha'}{2}} \int \mathrm{d}\xi^\pm \left(\alpha_0^\mu + \sum_{n\neq 0} \alpha_n^\mu e^{-\frac{\pi i}{l} n\xi^\pm}\right) \\ &= \frac{\pi}{l}\sqrt{\frac{\alpha'}{2}} \left(\alpha_0^\mu \xi^\pm - \frac{l}{\pi i} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-\frac{\pi i}{l} n\xi^\pm}\right) + \frac{x_0^\mu}{2}. \end{aligned} \tag{17}$$

Here, $\frac{x_0^\mu}{2}$ is just a suggestively named integration constant with no further meaning as yet. (It will later turn out to be the μ-coordinate of the string's $\sigma = 0$-end.) Inserting eq. (17) into our ansatz (14), we finally get the NN open string field's mode expansion

$$\begin{aligned} X_{\mathrm{NN}}^\mu(\xi^+, \xi^-) &= X_L^\mu(\xi^+) + X_L^\mu(\xi^-) \\ &= x_0^\mu + \frac{\pi}{l}\sqrt{\frac{\alpha'}{2}} \alpha_0^\mu (\xi^+ + \xi^-) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} \left(e^{-\frac{\pi i}{l} n\xi^+} + e^{-\frac{\pi i}{l} n\xi^-}\right) \\ &= x_0^\mu + \frac{\pi}{l}\sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-\frac{\pi i}{l} n\tau} \cos\left(\tfrac{\pi}{l} n\sigma\right). \end{aligned} \tag{18}$$

[1] If we were working with the auxiliary field $\hat{X}^\mu$, we could have arrived at the same conclusion by exploiting σ-parity in eq. (12), i.e. invariance of $\hat{X}^\mu$ under $\sigma \to -\sigma$.

c) Dirichlet boundary conditions require that the string endpoints are fixed, i.e. $\delta X^\mu_{\text{DD}}(\tau,\sigma) = 0$ for $\sigma \in \{0,l\}$, or equivalently $\partial_\tau X^\mu_{\text{DD}}(\tau,0) = \partial_\tau X^\mu_{\text{DD}}(\tau,l) = 0$. A suitable auxiliary field that automatically implements Dirichlet boundary conditions would be

$$\hat{X}_\mu(\tau,\sigma) = \begin{cases} X_\mu(\tau,\sigma) & \text{for } 0 \le \sigma \le l, \\ -X_\mu(\tau,-\sigma) & \text{for } -l \le \sigma \le 0. \end{cases} \tag{19}$$

Again, we prefer to proceed with the actual string field. Everything we did in part b) for the derivation of open NN string mode expansion carries over one-to-one up to eq. (13). Since we now differentiate w.r.t. τ, which appears with positive sign in both ξ^+ and ξ^-, we don't pick up a minus here from the chain rule when calculating $\partial_\tau X^\mu_{\text{DD}}(\xi^\pm)$. Consequently, we get

$$\partial_\tau X^\mu_{\text{DD}}(\tau,\sigma)\big|_{\sigma=0} = \partial_+ X^\mu_L(\tau) + \partial_- X^\mu_R(\tau) = 0, \tag{20}$$

which allows us to identify $X^\mu_R = -X^\mu_L + c^\mu$ and eq. (14) becomes

$$X^\mu_{\text{DD}}(\xi^+,\xi^-) = X^\mu_L(\xi^+) - X^\mu_L(\xi^-) + c^\mu. \tag{21}$$

Note that this time, the additive constant c^μ cannot be absorbed equally into $X^\mu_L(\xi^+)$ and $X^\mu_L(\xi^-)$ because they appear with different sign. Instead, by taking the equation to the $\sigma = 0$ endpoint of the string,

$$X^\mu_{\text{DD}}(\xi^+,\xi^-)\big|_{\sigma=0} = X^\mu_L(\tau) - X^\mu_L(\tau) + c^\mu = c^\mu \equiv x^\mu_0, \tag{22}$$

we find c^μ is now the fixed position x^μ_0 of one end of the string. At $\sigma = l$, we find

$$\partial_\tau X^\mu_{\text{DD}}(\tau,\sigma)\big|_{\sigma=l} = \partial_+ X^\mu_L(\tau+l) - \partial_- X^\mu_L(\tau-l) = 0, \tag{23}$$

i.e. as expected, $\partial_\pm X^\mu_L$ is again periodic and can be expanded into the same Fourier series (16) and integrated in the same way as before. Plugging in the integrated Fourier series (17) into $X^\mu_{\text{DD}}(\tau,\sigma)$ from eq. (21) yields the mode expansion for the DD open string,

$$\begin{aligned} X^\mu_{\text{DD}}(\xi^+,\xi^-) &= x^\mu_0 + X^\mu_L(\xi^+) - X^\mu_L(\xi^-) \\ &= x^\mu_0 + \frac{\pi}{l}\sqrt{\tfrac{\alpha'}{2}}\,\alpha^\mu_0(\xi^+ - \xi^-) + i\sqrt{\tfrac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}\left(e^{-\frac{\pi i}{l}n\xi^+} - e^{-\frac{\pi i}{l}n\xi^-}\right) \\ &= x^\mu_0 + \frac{\pi}{l}\sqrt{2\alpha'}\,\alpha^\mu_0\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}\sin\left(\tfrac{\pi}{l}n\sigma\right). \end{aligned} \tag{24}$$

At $\sigma = l$, this mode expansion reads

$$X^\mu_{\text{DD}}(\tau,l) = x^\mu_0 + \pi\sqrt{2\alpha'}\,\alpha^\mu_0, \tag{25}$$

which is indeed independent of τ as it must be to respect the fixed string end. From this we can gather that the string stretches across a distance of $\pi\sqrt{2\alpha'}\,\alpha^\mu_0$ in the μ-direction,

$$X^\mu_{\text{DD}}(\tau,l) - X^\mu_{\text{DD}}(\tau,0) = x^\mu_0 + \pi\sqrt{2\alpha'}\,\alpha^\mu_0 - x^\mu_0 = \pi\sqrt{2\alpha'}\,\alpha^\mu_0 \equiv x^\mu_l - x^\mu_0 \tag{26}$$

where it makes sense to define x^μ_l as the μ-coordinate of the other fixed point of the string in order to write the mode expansion as

$$X^\mu_{\text{DD}}(\xi^+,\xi^-) = x^\mu_0 + (x^\mu_l - x^\mu_0)\frac{\sigma}{l} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}\sin\left(\tfrac{\pi}{l}n\sigma\right). \tag{27}$$

d) DN boundary conditions translate into

$$\partial_\tau X^\mu_{\text{DN}}(\tau,\sigma)\big|_{\sigma=0} = 0 \qquad \text{and} \qquad \partial_\sigma X^\mu_{\text{DN}}(\tau,\sigma)\big|_{\sigma=l} = 0. \tag{28}$$

We therefore again get eq. (21) with the string's $\sigma = 0$-end fixed at x_0^μ

$$X^\mu_{\mathrm{DN}}(\xi^+,\xi^-) = X^\mu_L(\xi^+) - X^\mu_L(\xi^-) + x^\mu_0. \tag{29}$$

Applying the condition $X^\mu_{\mathrm{DN}}(\tau,\sigma)\big|_{\sigma=l} = 0$ gives

$$\partial_\sigma X^\mu_{\mathrm{DN}}(\tau,\sigma)\big|_{\sigma=l} = \partial_+ X^\mu_L(\tau+l) + \partial_- X^\mu_L(\tau-l) = 0, \tag{30}$$

which marks $\partial_+ X^\mu_L$ as an antiperiodic function of the form $f\big(t-\frac{T}{4}\big) = -f\big(t+\frac{T}{4}\big)$, where the period is now $T = 4l$. Doubling the period can be implemented in the Fourier expansion by making the mode counter half-integer, i.e.

$$\partial_\pm X^\mu_L(\xi^\pm) = \frac{\pi}{l}\sqrt{\tfrac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \alpha^\mu_n e^{-\frac{\pi i}{l} n \xi^\pm}. \tag{31}$$

One immediate consequence is that there is no longer a phase-factor-free zero mode α^μ_0. If we integrate eq. (31) and plug the result into eq. (29), we get

$$\begin{aligned} X^\mu_{\mathrm{DN}}(\xi^+,\xi^-) &= x^\mu_0 + X^\mu_L(\xi^+) - X^\mu_L(\xi^-) \\ &= x^\mu_0 + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\alpha^\mu_n}{n}\Big(e^{-\frac{\pi i}{l}n\xi^+} - e^{-\frac{\pi i}{l}n\xi^-}\Big) \\ &= x^\mu_0 - \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\alpha^\mu_n}{n} e^{-\frac{\pi i}{l}n\tau} \sin\big(\tfrac{\pi}{l}n\sigma\big). \end{aligned} \tag{32}$$

e) This business of finding mode expansions is getting dull. But fear not, we can change the topic now. Our next task is to find the centre-of-mass position q^μ and the total momentum p^μ of the string for each set of boundary conditions discussed above.

1. NN:

$$\begin{aligned} q^\mu_{\mathrm{NN}} &\equiv \frac{1}{l}\int_0^l \mathrm{d}\sigma\, X^\mu_{\mathrm{NN}}(\tau,\sigma) = \frac{1}{l}\int_0^l \mathrm{d}\sigma \Big(x^\mu_0 + \frac{\pi}{l}\sqrt{2\alpha'}\alpha^\mu_0\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n} e^{-\frac{\pi i}{l}n\tau}\cos\big(\tfrac{\pi}{l}n\sigma\big)\Big) \\ &= x^\mu_0 + \frac{\pi}{l}\sqrt{2\alpha'}\alpha^\mu_0\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n} e^{-\frac{\pi i}{l}n\tau}\frac{1}{l}\int_0^l \mathrm{d}\sigma\, \underbrace{\cos\big(\tfrac{\pi}{l}n\sigma\big)}_{\text{antisymmetric around } \sigma=\frac{l}{2}} \\ &= x^\mu_0 + \frac{\pi}{l}\sqrt{2\alpha'}\alpha^\mu_0\tau, \end{aligned} \tag{33}$$

$$\begin{aligned} p^\mu_{\mathrm{NN}} &= \int_0^l \mathrm{d}\sigma\, \Pi^\mu_{\mathrm{NN}}(\tau,\sigma) = T\int_0^l \mathrm{d}\sigma\, \partial_\tau X^\mu_{\mathrm{NN}}(\tau,\sigma) \\ &= T\int_0^l \mathrm{d}\sigma \Big(\frac{\pi}{l}\sqrt{2\alpha'}\alpha^\mu_0 + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha^\mu_n \frac{\pi}{l} e^{-\frac{\pi i}{l}n\tau}\cos\big(\tfrac{\pi}{l}n\sigma\big)\Big) \\ &= \pi T\sqrt{2\alpha'}\alpha^\mu_0 + T\sqrt{2\alpha'}\sum_{n\neq 0}\alpha^\mu_n\frac{\pi}{l}e^{-\frac{\pi i}{l}n\tau}\underbrace{\int_0^l \mathrm{d}\sigma\, \cos\big(\tfrac{\pi}{l}n\sigma\big)}_{0\text{, as above}} \\ &= \pi T\sqrt{2\alpha'}\alpha^\mu_0 = \frac{\alpha^\mu_0}{\sqrt{2\alpha'}}. \end{aligned} \tag{34}$$

Knowing that $\alpha^\mu_0 = \sqrt{2\alpha'}p^\mu_{\mathrm{NN}}$, we can also write the open NN string's centre-of-mass position as

$$q^\mu_{\mathrm{NN}} = x^\mu_0 + \frac{2\pi\alpha'}{l}p^\mu_{\mathrm{NN}}\tau = x^\mu_0 + \frac{p^\mu_{\mathrm{NN}}}{Tl}\,\tau. \tag{35}$$

This is a very nice result as it means that the string's c.o.m. just moves in a straight line with constant velocity $\frac{p^\mu_{\mathrm{NN}}}{Tl}$. This is consistent with the fact that so far we've been working entirely in a free theory (see Polyakov action, no terms above quadratic order appear).

2. DD:

$$\begin{aligned}
q^\mu_{\mathrm{DD}} &= \frac{1}{l}\int_0^l \mathrm{d}\sigma\, X^\mu_{\mathrm{DD}}(\tau,\sigma) = \frac{1}{l}\int_0^l \mathrm{d}\sigma \Big(x^\mu_0 + (x^\mu_l - x^\mu_0)\frac{\sigma}{l} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n} e^{-\frac{\pi i}{l}n\tau}\sin\left(\tfrac{\pi}{l}n\sigma\right)\Big) \\
&= x^\mu_0 + \frac{1}{2}(x^\mu_l - x^\mu_0) + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}\frac{1}{l}\int_0^l \mathrm{d}\sigma\,\sin\left(\tfrac{\pi}{l}n\sigma\right) \\
&= \frac{1}{2}(x^\mu_0 + x^\mu_l) + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}\frac{1}{\pi n}\big(1 - \underbrace{\cos(\pi n)}_{(-1)^n}\big) \\
&= \frac{1}{2}(x^\mu_0 + x^\mu_l) + \frac{2}{\pi}\sqrt{2\alpha'}\sum_{\substack{n\in 2\mathbb{Z}\\ n\neq 0}}\frac{\alpha^\mu_n}{n^2}e^{-\frac{\pi i}{l}n\tau},
\end{aligned} \tag{36}$$

$$\begin{aligned}
p^\mu_{\mathrm{DD}} &= \int_0^l \mathrm{d}\sigma\,\Pi^\mu_{\mathrm{DD}}(\tau,\sigma) = T\int_0^l \mathrm{d}\sigma\,\partial_\tau X^\mu_{\mathrm{DD}}(\tau,\sigma) \\
&= T\int_0^l \mathrm{d}\sigma\Big(-\frac{\pi i}{l}\sqrt{2\alpha'}\sum_{n\neq 0}\alpha^\mu_n e^{-\frac{\pi i}{l}n\tau}\sin\left(\tfrac{\pi}{l}n\sigma\right)\Big) \\
&= -T\frac{\pi i}{l}\sqrt{2\alpha'}\sum_{n\neq 0}\alpha^\mu_n\frac{\pi}{l}e^{-\frac{\pi i}{l}n\tau}\frac{l}{\pi n}\big(1 - \underbrace{\cos(\pi n)}_{(-1)^n}\big) = \frac{\sqrt{2/\alpha'}}{il}\sum_{\substack{n\in 2\mathbb{Z}\\ n\neq 0}}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}.
\end{aligned} \tag{37}$$

Interestingly, both c.o.m. position and total momentum depend on even-numbered modes α^μ_n with $n \in 2\mathbb{Z}$.

3. DN:

$$\begin{aligned}
q^\mu_{\mathrm{DN}} &= \frac{1}{l}\int_0^l \mathrm{d}\sigma\, X^\mu_{\mathrm{DN}}(\tau,\sigma) = \frac{1}{l}\int_0^l \mathrm{d}\sigma\Big(x^\mu_0 - \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}\sin\left(\tfrac{\pi}{l}n\sigma\right)\Big) \\
&\overset{(36)}{=} x^\mu_0 - \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}\frac{1}{\pi n}\big(1 - \underbrace{\cos(\pi n)}_{0}\big) = x^\mu_0 - \frac{\sqrt{2\alpha'}}{\pi}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{\alpha^\mu_n}{n^2}e^{-\frac{\pi i}{l}n\tau},
\end{aligned} \tag{38}$$

$$\begin{aligned}
p^\mu_{\mathrm{DN}} &= \int_0^l \mathrm{d}\sigma\,\Pi^\mu_{\mathrm{DN}}(\tau,\sigma) = T\int_0^l \mathrm{d}\sigma\,\partial_\tau X^\mu_{\mathrm{DN}}(\tau,\sigma) \\
&\overset{(37)}{=} T\frac{\pi i}{l}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha^\mu_n\frac{\pi}{l}e^{-\frac{\pi i}{l}n\tau}\frac{l}{\pi n}(1-0) = \frac{i\sqrt{2/\alpha'}}{l}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}.
\end{aligned} \tag{39}$$

2 Conformal transformations and their algebra

The conformal Killing equation is defined as

$$(P\cdot\epsilon)_{ab} \equiv \nabla_a\epsilon_b + \nabla_b\epsilon_a - h_{ab}\nabla^c\epsilon_c = 0. \tag{40}$$

a) Recall the significance of a vector field $\epsilon(\xi)$ satisfying the conformal Killing equation (40).

b) Show that a conformal Killing vector leads to a conserved current $J^a = T^{ab}\epsilon_b$. Which properties of T^{ab} are needed for this?

c) Now go to lightcone gauge and show from the definition in part a) that the conformal Killing vectors are precisely the ones generating transformations

$$\xi^\pm \to \tilde{\xi}^\pm(\xi^\pm) = \xi^\pm + \epsilon^\pm(\xi^\pm). \tag{41}$$

Recall from the lecture that these were the residual gauge symmetries in flat gauge.

d) Using $T_{\pm\pm} = -\frac{1}{\alpha'}\partial_\pm X \cdot \partial_\pm X$ and the equal-time Poisson brackets

$$\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{\text{PB}} = \{\dot{X}^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')\}_{\text{PB}} = 0, \tag{42}$$

$$\{X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')\}_{\text{PB}} = \frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma'), \tag{43}$$

calculate the Poisson brackets

$$\{T_{\pm\pm}(\tau,\sigma), X^\mu(\tau,\sigma')\}_{\text{PB}}. \tag{44}$$

Hint: It is useful to first work out, from the definition of the Poisson bracket as given in the lecture, how to express quantities such as $\{AB, C\}$ in terms of elementary Poisson brackets involving only two fields.

e) Use the definition (for the closed string)

$$L_{\epsilon^\pm} \equiv -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma\, \epsilon^\pm(\xi^\pm)T_{\pm\pm}(\xi^\pm), \tag{45}$$

and the result of part d) to calculate the Poisson brackets

$$\{L_{\epsilon^\pm}, X^\mu(\tau,\sigma)\}_{\text{PB}}. \tag{46}$$

Use this to argue that the $L_{\epsilon^\pm}$ generate infinitesimal conformal transformations via the Poisson bracket.

f) One can decompose the functions $\epsilon^\pm(\xi^\pm)$ into a discrete sum of Fourier components $e^{i\frac{2\pi}{l}m\xi^\pm}$, $m \in \mathbb{Z}$. The resulting generators

$$L_m^- = -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma\, T_{--}e^{i\frac{2\pi}{l}m(\tau-\sigma)}, \qquad L_m^+ = -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma\, T_{++}e^{i\frac{2\pi}{l}m(\tau+\sigma)}, \tag{47}$$

then form two copies of the Witt algebra with respect to the Poisson bracket, i.e.

$$\{L_m^\pm, L_n^\pm\} = -i(m-n)L_{m+n}^\pm. \tag{48}$$

Verify explicitly that the above commutation relations satisfy the Jacobi identity, i.e. form a Lie algebra.

g) Show that the generators L_0, L_1 and L_{-1} form a Lie subalgebra.

h) Show that the combination $(L_0^+ - L_0^-)$ generates rigid σ-translations along the closed string. How about $(L_0^+ + L_0^-)$?

a) A long list could be compiled of all the important properties of conformal Killing vector fields. We will mention only three.

1. Vector fields that satisfy the conformal Killing equation are exactly those whose flow preserves the conformal structure of a manifold. Expressed in the language of conformal geometry: The conformal Killing equation on a manifold M with metric tensor $\boldsymbol{h}$ applies to those vector fields $\epsilon(\xi)$ which preserve $\boldsymbol{h}$ up to a scaling, i.e.

$$\mathcal{L}_\epsilon \boldsymbol{h} = \lambda(\xi)\, \boldsymbol{h} \tag{49}$$

where $\mathcal{L}_\epsilon$ is the Lie derivative and $\lambda(\xi)$ some function of position on M. It is easy to see, that eq. (49) is completely equivalent to eq. (40). Inserting $\mathcal{L}_\epsilon h_{ab} = \nabla_a\epsilon_b + \nabla_b\epsilon_a$ into eq. (40)

and bringing $h_{ab}\nabla^c\epsilon_c$ to the other side, we have

$$\mathcal{L}_\epsilon h_{ab} = \nabla_a\epsilon_b + \nabla_b\epsilon_a = (\nabla^c\epsilon_c)\, h_{ab}. \tag{50}$$

Hence $\lambda(\xi)$ is given by $\nabla^c\epsilon_c$.

2. In particular, property 1 results in the fact that for diffeomorphisms of the form (41), i.e.

$$\xi^a \to \tilde{\xi}^a(\xi) = \xi^a + \epsilon^a(\xi), \qquad \xi^a \in \{\tau,\sigma\}, \tag{51}$$

where $\epsilon^a(\xi)$ is a conformal Killing vector field, the effect on the metric can be undone by a Weyl rescaling.

3. Even after gauge fixing to the flat metric, $h_{ab} = \eta_{ab}$, the Polyakov action retains a large residual gauge symmetry, of which the conformal Killing vectors are the generators.

A fourth very important property will be demonstrated in part b).

b) Every conformal Killing vector field $\epsilon^a(\xi)$ yields an associated conserved current $J^a_\epsilon = T^{ab}\epsilon_b$ with $\nabla_a J^a_\epsilon = 0$. There exist infinitely many such $\epsilon^a(\xi)$.

Proof: By Noether's theorem, T_{ab} is the conserved current resulting from local diffeomorphism invariance on the worldsheet. In math: $\nabla^a T_{ab} = 0$ for an on-shell string field $X(\tau,\sigma)$ since the action remains invariant under the following change in coordinates $\xi^a \in \{\tau,\sigma\}$ and metric h_{ab},

$$\xi^a \to \tilde{\xi}^a(\xi) = \xi^a + \epsilon^a(\xi), \tag{52}$$

$$h_{ab} \to h'_{ab} = h_{ab} + \delta h_{ab} = h_{ab} + \nabla_a\epsilon_b + \nabla_b\epsilon_a. \tag{53}$$

To make this statement explicit, we recall the definition of the energy-momentum i.t.o. the Polyakov action,

$$T_{ab} = \frac{4\pi}{\sqrt{-h}}\frac{\delta S_\text{P}}{\delta h^{ab}}, \tag{54}$$

followed by considering the total variation of $S_\text{P} = S_\text{P}[X,h]$,

$$0 = \delta S_\text{P} = \int \mathrm{d}\tau\mathrm{d}\sigma\, \underbrace{\frac{\delta S_\text{P}}{\delta X^\mu}}_{0}\, \delta X^\mu + \int \mathrm{d}\tau\mathrm{d}\sigma\, \frac{\delta S_\text{P}}{\delta h^{ab}}\, \delta h^{ab} \overset{(54)}{=} \frac{1}{4\pi}\int \mathrm{d}\tau\mathrm{d}\sigma\, \sqrt{-h}\, T_{ab}\, h^{ab} \tag{55}$$

where the very first equality holds because the action is invariant under the transformations (52) and (53), and the variation of S_P w.r.t. X^μ vanishes on-shell, i.e. upon use of the string field's e.o.m. Inserting δh_{ab} as given in eq. (53) and using $T^{ab} = T^{ba}$, we get

$$\begin{aligned} 0 &= \frac{1}{4\pi}\int \mathrm{d}\tau\mathrm{d}\sigma\, \sqrt{-h}\, T_{ab}\, h^{ab} = \frac{1}{2\pi}\int \mathrm{d}\tau\mathrm{d}\sigma\, \sqrt{-h}\, T_{ab}\, \nabla^a\epsilon^b \\ &= \frac{1}{2\pi}\int \mathrm{d}\tau\mathrm{d}\sigma \left[\nabla^a\Big(\sqrt{-h}\, T_{ab}\,\epsilon^b\Big) - \nabla^a\Big(\sqrt{-h}\, T_{ab}\Big)\epsilon^b\right] \\ &= -\frac{1}{2\pi}\int \mathrm{d}\tau\mathrm{d}\sigma\, \nabla^a\left(\sqrt{-h}\, T_{ab}\right)\epsilon^b, \end{aligned} \tag{56}$$

where the first term in the second line is a volume integral over a total derivative. Using Stokes theorem, it can be converted to a surface integral sans derivative, which by the usual argument of exclusively treating localized systems that do not extend outwards to infinity, can be taken to vanish. The remaining term in the third line has to vanish for all and every $\epsilon^b(\xi)$ since the action is invariant under general diffeomorphisms. The only way for the integral to be zero regardless of which $\epsilon^b(\xi)$ we plug in, is for $\nabla^a(\sqrt{-h}\, T_{ab})$ to be zero. From $\nabla^a h_{bc} = 0$, it follows by the chain rule that also $\nabla^a\sqrt{-h} = 0$. Hence we are left with our initial statement,

$$\nabla^a T_{ab} = 0. \tag{57}$$

Due to eq. (57), we have

$$\nabla_a J^a_\epsilon = \nabla_a (T^{ab}\epsilon_b) = T^{ab}\nabla_a \epsilon_b = \frac{1}{2}T^{ab}(\nabla_a\epsilon_b + \nabla_b\epsilon_a), \tag{58}$$

where in the last step, we again exploited the energy-momentum tensor's symmetry $T^{ab} = T^{ba}$. Inserting the conformal Killing equation, we get

$$\nabla_a J^a_\epsilon = \frac{1}{2}\underbrace{T^{ab}h_{ab}}_{0}\nabla^c\epsilon_c = 0, \tag{59}$$

where we used our result from exercise 1.f) on assignment sheet 2 that the energy-momentum is traceless in any system with an action invariant under Weyl rescalings.

c) The term lightcone gauge can be confusing if one assumes the word lightcone to describe the gauge. A more fitting name for this modus operandi might be flat gauge in lightcone coordinates. What is actually happening is that we partially fix the gauge by flattening the metric, $h_{ab} \overset{!}{=} \eta_{ab}$, followed by a transformation into lightcone coordinates $\xi^\pm = \tau \pm \sigma$, where now the metric components a, b are no longer $\in \{\tau, \sigma\}$ but $\in \{+, -\}$. Its components can be inferred from the line element $\mathrm{d}s^2$:

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}\sigma^2 = -\mathrm{d}\xi^+\mathrm{d}\xi^- = \eta_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b \quad \overset{\eta_{ab}=\eta_{ba}}{\Rightarrow} \quad \boldsymbol{\eta} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{60}$$

Since in flat space $\nabla_a = \partial_a$, the conformal Killing equation in lightcone gauge reads

$$\partial_a\epsilon_b + \partial_b\epsilon_a = \eta_{ab}\partial_c\epsilon^c. \tag{61}$$

From this, we get three equations, namely

$$\partial_+\epsilon_- + \partial_-\epsilon_+ = -\frac{1}{2}(\partial_+\epsilon^+ + \partial_-\epsilon^-), \qquad \partial_+\epsilon_+ = 0, \qquad \partial_-\epsilon_- = 0. \tag{62}$$

The first one is uninteresting, but inserting $\epsilon_\pm = \eta_{\pm\mp}\epsilon^\mp = -\frac{1}{2}\epsilon^\mp$ into the second and third, respectively, we get

$$\partial_+\epsilon^- = 0, \qquad \partial_-\epsilon^+ = 0. \tag{63}$$

From eq. (63), we learn that $\epsilon^\pm$ are functions of $\xi^\pm$ only, i.e. $\epsilon^+ = \epsilon^+(\xi^+)$ and $\epsilon^- = \epsilon^-(\xi^-)$. We have thus reproduced the coordinate dependencies in eq. (41).

Again we note that a transformation of the form

$$\xi^\pm \to \tilde{\xi}^\pm(\xi^\pm) = \xi^\pm + \epsilon^\pm(\xi^\pm). \tag{64}$$

is not prohibited even after partially fixing the gauge freedom by setting $h_{ab} \overset{!}{=} \eta_{ab}$. This remaining invariance thus constitutes the residual gauge symmetry in flat gauge.

d) The part of Noether's theorem stating that continuous symmetries give rise to conserved currents and associated charges is well known. What is sometimes overlooked is the fact, that Noether's theorem works both ways. In this part, we will demonstrate the converse, namely that conserved charges also generate symmetries. More precisely, we show that the stress energy tensor generates conformal transformations.

To do so, we will make heavy use of the (equal-time) Poisson bracket defined for two fields $F(\tau,\sigma)$, $G(\tau,\sigma')$ as

$$\{F(\tau,\sigma), G(\tau,\sigma')\}_{\mathrm{PB}} \equiv \int \mathrm{d}\tilde{\sigma}\left(\frac{\partial F(\tau,\sigma)}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial G(\tau,\sigma')}{\partial \Pi_\mu(\tau,\tilde{\sigma})} - \frac{\partial G(\tau,\sigma')}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial F(\tau,\sigma)}{\partial \Pi_\mu(\tau,\tilde{\sigma})}\right). \tag{65}$$

To proceed, we employ the identity $\{FG,H\}_{PB} = F\{G,H\}_{PB} + \{F,H\}_{PB}G$, where F, G, and H are each fields. Using the product rule, its validity is easy to see,

$$\begin{aligned}
&\{F(\tau,\sigma)G(\tau,\sigma'),H(\tau,\sigma'')\}_{\mathrm{PB}}\\
&\quad=\int \mathrm{d}\tilde{\sigma}\left(\frac{\partial[F(\tau,\sigma)G(\tau,\sigma')]}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial H(\tau,\sigma'')}{\partial \Pi_\mu(\tau,\tilde{\sigma})}-\frac{\partial H(\tau,\sigma'')}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial[F(\tau,\sigma)G(\tau,\sigma')]}{\partial \Pi_\mu(\tau,\tilde{\sigma})}\right)\\
&\quad=F(\tau,\sigma)\int \mathrm{d}\tilde{\sigma}\left(\frac{\partial G(\tau,\sigma')}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial H(\tau,\sigma'')}{\partial \Pi_\mu(\tau,\tilde{\sigma})}-\frac{\partial H(\tau,\sigma'')}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial G(\tau,\sigma')}{\partial \Pi_\mu(\tau,\tilde{\sigma})}\right)\\
&\qquad+\int \mathrm{d}\tilde{\sigma}\left(\frac{\partial F(\tau,\sigma)}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial H(\tau,\sigma'')}{\partial \Pi_\mu(\tau,\tilde{\sigma})}-\frac{\partial H(\tau,\sigma'')}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial F(\tau,\sigma)}{\partial \Pi_\mu(\tau,\tilde{\sigma})}\right)G(\tau,\sigma')\\
&\quad=F(\tau,\sigma)\{G(\tau,\sigma'),H(\tau,\sigma'')\}_{\mathrm{PB}}+\{F(\tau,\sigma),H(\tau,\sigma'')\}_{\mathrm{PB}}G(\tau,\sigma').
\end{aligned} \tag{66}$$

With this and the relations given in the exercise, which we restate here for convenience,

$$T_{\pm\pm} = -\frac{1}{\alpha'}\partial_\pm X^\mu \partial_\pm X_\mu, \tag{67}$$

$$\{X^\mu(\tau,\sigma),X^\nu(\tau,\sigma')\}_{\mathrm{PB}} = \{\dot{X}^\mu(\tau,\sigma),\dot{X}^\nu(\tau,\sigma')\}_{\mathrm{PB}} = 0, \tag{68}$$

$$\{X^\mu(\tau,\sigma),\dot{X}^\nu(\tau,\sigma')\}_{\mathrm{PB}} = \frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{69}$$

we can simplify the Poisson bracket in eq. (44),

$$\begin{aligned}
\{T_{\pm\pm}(\tau,\sigma),X^\mu(\tau,\sigma')\}_{\mathrm{PB}} &\overset{(67)}{=} -\frac{1}{\alpha'}\{\partial_\pm X_\nu(\tau,\sigma)\partial_\pm X^\nu(\tau,\sigma),X^\mu(\tau,\sigma')\}_{\mathrm{PB}}\\
&\overset{(66)}{=} -\frac{2}{\alpha'}\partial_\pm X_\nu(\tau,\sigma)\{\partial_\pm X^\nu(\tau,\sigma),X^\mu(\tau,\sigma')\}_{\mathrm{PB}}\\
&= -\frac{1}{\alpha'}\partial_\pm X_\nu(\tau,\sigma)\Big[\{\partial_\tau X^\nu(\tau,\sigma),X^\mu(\tau,\sigma')\}_{\mathrm{PB}}\\
&\qquad\qquad \pm\{\partial_\sigma X^\nu(\tau,\sigma),X^\mu(\tau,\sigma')\}_{\mathrm{PB}}\Big],
\end{aligned} \tag{70}$$

where in the last step we used $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. In the last term of eq. (70), ∂_σ does not act upon $X^\mu(\tau,\sigma')$, so we may as well pull it out of the bracket and use eq. (68) to obtain

$$\{\partial_\sigma X^\nu(\tau,\sigma),X^\mu(\tau,\sigma')\}_{\mathrm{PB}} = \partial_\sigma\{X^\nu(\tau,\sigma),X^\mu(\tau,\sigma')\}_{\mathrm{PB}} \overset{(68)}{=} 0. \tag{71}$$

For the term $\{\partial_\tau X^\nu(\tau,\sigma),X^\mu(\tau,\sigma')\}_{\mathrm{PB}}$, we can insert eq. (69) to arrive at

$$\{T_{\pm\pm}(\tau,\sigma),X^\mu(\tau,\sigma')\}_{\mathrm{PB}} = \frac{1}{\alpha' T}\partial_\pm X_\nu(\tau,\sigma)\eta^{\mu\nu}\delta(\sigma-\sigma') = 2\pi\partial_\pm X^\mu(\tau,\sigma)\delta(\sigma-\sigma'). \tag{72}$$

e) With eq. (72) in our toolbox, it is trivial to calculate the Poisson bracket of the string field $X^\mu(\tau,\sigma)$ and the conserved charges $L_{\epsilon^\pm}$ associated with invariance under conformal Killing transformation. The $L_{\epsilon^\pm}$ are given by

$$L_{\epsilon^\pm} = -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma\,\epsilon^\pm(\xi^\pm)T_{\pm\pm}(\xi^\pm). \tag{73}$$

Therefore,

$$\begin{aligned}
\{L_{\epsilon^\pm},X^\mu(\tau,\sigma)\}_{\mathrm{PB}} &= -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma'\,\epsilon^\pm(\xi^{\pm\prime})\,\{T_{\pm\pm}(\tau,\sigma'),X^\mu(\tau,\sigma)\}_{\mathrm{PB}}\\
&\overset{(72)}{=} -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma'\,\epsilon^\pm(\xi^{\pm\prime})\,2\pi\partial_\pm X^\mu(\tau,\sigma')\delta(\sigma-\sigma')\\
&= -\frac{l}{2\pi}\epsilon^\pm(\xi^\pm)\partial_\pm X^\mu(\tau,\sigma).
\end{aligned} \tag{74}$$

Since this is precisely the Lie derivative acting on the string field, i.e.

$$-\frac{2\pi}{l}\{L_{\epsilon^\pm},X^\mu(\tau,\sigma)\}_{\mathrm{PB}} = \epsilon^\pm(\xi^\pm)\partial_\pm X^\mu(\tau,\sigma) = \mathcal{L}_{\epsilon^\pm}X^\mu(\tau,\sigma), \tag{75}$$

we have shown that the $L_{\epsilon^\pm}$ generate conformal transformations.

f) By decomposing a Killing vector field $\epsilon^\pm(\xi^\pm)$ into its Fourier components, i.e.

$$\epsilon^\pm(\xi^\pm) = \sum_{m\in\mathbb{Z}} \epsilon^\pm_m \, e^{i\frac{2\pi}{l}m\xi^\pm}, \tag{76}$$

and inserting this sum into the conserved charges $L_{\epsilon^\pm}$, we get a series representation for the charges themselves,

$$L_{\epsilon^\pm} = -\frac{l}{4\pi^2}\sum_{m\in\mathbb{Z}}\int_0^l \mathrm{d}\sigma\, \epsilon^\pm_m \, T_{\pm\pm}(\xi^\pm)\, e^{i\frac{2\pi}{l}m\xi^\pm} \equiv \sum_{m\in\mathbb{Z}} \epsilon^\pm_m \, L^\pm_m. \tag{77}$$

In eq. (77), the $L^\pm_m$ are the generators of the conformal Killing transformations. This assertion will be demonstrated in part h). The $L^\pm_m$ are called Virasoro generators and will turn out to be of vital importance as a tool in the full quantum theory to ensure that our description is free of unitarity-spoiling ghosts. Note that eq. (77) establishes the Virasoro generators as the Fourier modes of the energy-momentum tensor T_{ab}. As stated in eq. (47), they are given by[2]

$$L^\pm_m = -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma\, T_{\pm\pm}\, e^{i\frac{2\pi}{l}m\xi^\pm}. \tag{78}$$

Although we were not tasked to do so by the exercise, we will verify that the set of $L^\pm_m$, $m \in \mathbb{Z}$ do indeed satisfy the classical Witt algebra (48). To that end, we calculate the Poisson bracket of the energy-momentum tensor in lightcone coordinates with itself,

$$\begin{aligned}
\big\{T_{\pm\pm}(\tau,\sigma), T_{\pm\pm}(\tau,\sigma')\big\}_{\mathrm{PB}} &= \frac{1}{\alpha'^2}\Big\{\partial_\pm X_\mu(\tau,\sigma)\partial_\pm X^\mu(\tau,\sigma), \partial_\pm X_\nu(\tau,\sigma')\partial_\pm X^\nu(\tau,\sigma')\Big\}_{\mathrm{PB}} \\
&\overset{(66)}{=} \frac{4}{\alpha'^2}\partial_\pm X_\mu(\tau,\sigma)\partial_\pm X_\nu(\tau,\sigma')\Big\{\partial_\pm X^\mu(\tau,\sigma), \partial_\pm X^\nu(\tau,\sigma')\Big\}_{\mathrm{PB}} \qquad [\partial_\pm = \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma)] \\
&= \frac{1}{\alpha'^2}\partial_\pm X_\mu(\tau,\sigma)\partial_\pm X_\nu(\tau,\sigma')\Big[\underbrace{\{\partial_\tau X^\mu(\tau,\sigma), \partial_\tau X^\nu(\tau,\sigma')\}_{\mathrm{PB}}}_{0,\text{ by eq. }(68)} \pm \underbrace{\{\partial_\tau X^\mu(\tau,\sigma), \partial_{\sigma'} X^\nu(\tau,\sigma')\}_{\mathrm{PB}}}_{-\partial_{\sigma'}\{X^\nu(\tau,\sigma'),\partial_\tau X^\mu(\tau,\sigma)\}_{\mathrm{PB}}} \\
&\qquad \pm \underbrace{\{\partial_\sigma X^\mu(\tau,\sigma), \partial_\tau X^\nu(\tau,\sigma')\}_{\mathrm{PB}}}_{\partial_\sigma\{X^\mu(\tau,\sigma),\partial_\tau X^\nu(\tau,\sigma')\}_{\mathrm{PB}}} + \underbrace{\{\partial_\sigma X^\mu(\tau,\sigma), \partial_{\sigma'} X^\nu(\tau,\sigma')\}_{\mathrm{PB}}}_{0,\text{ by eq. }(68)}\Big] \\
&\overset{(69)}{=} \frac{1}{\alpha'^2}\partial_\pm X_\mu(\tau,\sigma)\partial_\pm X_\nu(\tau,\sigma')\Big[\mp\partial_{\sigma'}\frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma') \pm \partial_\sigma\frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma')\Big] \\
&= \pm\frac{2\pi}{\alpha'}\partial_\pm X_\mu(\tau,\sigma)\partial_\pm X^\mu(\tau,\sigma')\Big[\partial_\sigma - \partial_{\sigma'}\Big]\delta(\sigma-\sigma').
\end{aligned} \tag{79}$$

For the Poisson bracket of the Virasoro generators themselves, we have

$$\begin{aligned}
\{L^\pm_m, L^\pm_n\}_{\mathrm{PB}} &\overset{(78)}{=} \frac{l^2}{(4\pi^2)^2}\int_0^l \mathrm{d}\sigma\int_0^l \mathrm{d}\sigma'\,\{T_{\pm\pm}(\tau,\sigma), T_{\pm\pm}(\tau,\sigma')\}_{\mathrm{PB}} e^{i\frac{2\pi}{l}m\xi^\pm} e^{i\frac{2\pi}{l}n\xi^{\pm\prime}} \\
&\overset{(79)}{=} \frac{l^2}{(2\pi)^4}\int_0^l \mathrm{d}\sigma\int_0^l \mathrm{d}\sigma'\left(\pm\frac{2\pi}{\alpha'}\partial_\pm X_\mu(\tau,\sigma)\partial_\pm X^\mu(\tau,\sigma')\delta(\sigma-\sigma')\Big[\partial_\sigma - \partial_{\sigma'}\Big]\right) e^{i\frac{2\pi}{l}m\xi^\pm} e^{i\frac{2\pi}{l}n\xi^{\pm\prime}} \\
&= \pm\frac{l^2}{(2\pi)^3}\int_0^l \mathrm{d}\sigma\,\frac{1}{\alpha'}\partial_\pm X_\mu(\tau,\sigma)\partial_\pm X^\mu(\tau,\sigma')\delta(\sigma-\sigma')\Big[\pm i\tfrac{2\pi}{l}m \mp i\tfrac{2\pi}{l}n\Big] e^{i\frac{2\pi}{l}m\xi^\pm} e^{i\frac{2\pi}{l}n\xi^{\pm\prime}} \\
&= i\tfrac{2\pi}{l}(m-n)\frac{l^2}{(2\pi)^3}\int_0^l \mathrm{d}\sigma\,\underbrace{\frac{1}{\alpha'}\partial_\pm X_\mu(\tau,\sigma)\partial_\pm X^\mu(\tau,\sigma)}_{-T_{\pm\pm}(\tau,\sigma)} e^{i\frac{2\pi}{l}(m+n)\xi^\pm} \\
&= -i(m-n)\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma\, T_{\pm\pm}(\tau,\sigma) e^{i\frac{2\pi}{l}(m+n)\xi^\pm} = -i(m-n)\,L^\pm_{m+n}.
\end{aligned} \tag{80}$$

[2]Note that the lecture notes use a different notation here: $L^+_m \equiv \tilde{L}_m$ and $L^-_m \equiv L_m$.

Using eq. (80), it is easy to show that the Virasoro generators together with the Poisson bracket as binary operation fulfill the Jacobi identity.

$$
\begin{aligned}
&\{\{L_l^\pm, L_m^\pm\}_{\mathrm{PB}}, L_n^\pm\}_{\mathrm{PB}} + \{\{L_m^\pm, L_n^\pm\}_{\mathrm{PB}}, L_l^\pm\}_{\mathrm{PB}} + \{\{L_n^\pm, L_l^\pm\}_{\mathrm{PB}}, L_m^\pm\}_{\mathrm{PB}} \\
&\quad\overset{(80)}{=} -i(l-m)\{L_{l+m}^\pm, L_n^\pm\}_{\mathrm{PB}} - i(m-n)\{L_{m+n}^\pm, L_l^\pm\}_{\mathrm{PB}} - i(n-l)\{L_{n+l}^\pm, L_m^\pm\}_{\mathrm{PB}} \\
&\quad\overset{(80)}{=} -i[-i(l-m)(l+m-n) - i(m-n)(m+n-l) - i(n-l)(n+l-m)]\, L_{l+m+n}^\pm \\
&\quad = -[\underline{l^2} - \underline{\underline{m^2}} - n(l-m) + \underline{\underline{m^2}} - \utilde{n^2} - l(m-n) + \utilde{n^2} - \underline{l^2} - m(n+l)]\, L_{l+m+n}^\pm \\
&\quad = [\underline{nl} - \underline{\underline{nm}} + \utilde{lm} - \underline{ln} + \underline{\underline{mn}} + \utilde{ml}]\, L_{l+m+n}^\pm = 0.
\end{aligned}
\tag{81}
$$

Since the Poisson bracket is further by bilinear and antisymmetric under exchange of its arguments, it together with the Virasoro generators forms a Lie algebra, i.e. the Witt algebra is a Lie algebra.

g) If a set of generators form a Lie algebra, then a subset of those generators is called a Lie subalgebra, if the subset is closed under the Lie bracket. The subset $K = \{L_0^\pm, L_k^\pm, L_{-k}^\pm\}$ of all Virasoro generators forms a Lie subalgebra of the Witt algebra since

$$
\begin{aligned}
\{L_0^\pm, L_k^\pm\}_{\mathrm{PB}} &= -i(-k)L_k^\pm && \in K, \\
\{L_0^\pm, L_{-k}^\pm\}_{\mathrm{PB}} &= -ikL_{-k}^\pm && \in K, \\
\{L_k^\pm, L_{-k}^\pm\}_{\mathrm{PB}} &= -i2kL_0^\pm && \in K.
\end{aligned}
\tag{82}
$$

In particular, this holds for $k = 1$.

h) To compute the effect of $(L_0^+ - L_0^-)$ and $(L_0^+ + L_0^-)$ on the string field $X^\mu(\tau, \sigma)$, we first need to calculate the Poisson bracket of a Virasoro generator with $X^\mu(\tau, \sigma)$.

$$
\begin{aligned}
\left\{L_m^\pm, X^\mu(\tau,\sigma)\right\}_{\mathrm{PB}} &\overset{(78)}{=} -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma' \left\{T_{\pm\pm}(\tau,\sigma'), X^\mu(\tau,\sigma)\right\}_{\mathrm{PB}} e^{i\frac{2\pi}{l}m\xi^\pm} \\
&\overset{(72)}{=} -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma'\, 2\pi\partial_\pm X^\mu(\tau,\sigma')\delta(\sigma-\sigma')e^{i\frac{2\pi}{l}m\xi^\pm} \\
&= -\frac{l}{2\pi}\, e^{i\frac{2\pi}{l}m\xi^\pm}\,\partial_\pm X^\mu(\tau,\sigma).
\end{aligned}
\tag{83}
$$

With this identity, it is a simple matter to compute the Poisson brackets of $(L_0^+ - L_0^-)$ and $(L_0^+ + L_0^-)$ with $X^\mu(\tau, \sigma)$.

$$
\begin{aligned}
\left\{(L_0^+ - L_0^-), X^\mu(\tau,\sigma)\right\}_{\mathrm{PB}} &= \left\{L_0^+, X^\mu(\tau,\sigma)\right\}_{\mathrm{PB}} - \left\{L_0^-, X^\mu(\tau,\sigma)\right\}_{\mathrm{PB}} \\
&\overset{(83)}{=} -\frac{l}{2\pi}\partial_+ X^\mu(\tau,\sigma) + \frac{l}{2\pi}\partial_- X^\mu(\tau,\sigma) = -\frac{l}{2\pi}\partial_\sigma X^\mu(\tau,\sigma),
\end{aligned}
\tag{84}
$$

$$
\left\{(L_0^+ + L_0^-), X^\mu(\tau,\sigma)\right\}_{\mathrm{PB}} \overset{(83)}{=} -\frac{l}{2\pi}\partial_+ X^\mu(\tau,\sigma) - \frac{l}{2\pi}\partial_- X^\mu(\tau,\sigma) = -\frac{l}{2\pi}\partial_\tau X^\mu(\tau,\sigma),
\tag{85}
$$

where we used $\partial_+ - \partial_- = \frac{1}{2}(\partial_\tau + \partial_\sigma) - \frac{1}{2}(\partial_\tau - \partial_\sigma) = \partial_\sigma$ and $\partial_+ + \partial_- = \frac{1}{2}(\partial_\tau + \partial_\sigma) + \frac{1}{2}(\partial_\tau - \partial_\sigma) = \partial_\tau$. We have thus shown that $(L_0^+ - L_0^-)$ and $(L_0^+ + L_0^-)$ generate infinitesimal σ- and τ-translations, respectively.