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Quantum Field Theory II - Assignment 3

Problem 3.1 (Wick's theorem reloaded)

In the lecture, we have defined

$$\langle \phi(x_1)\dots\phi(x_n)\rangle_0 := e^{\frac{1}{2}\frac{\delta}{\delta\phi}D_F\frac{\delta}{\delta\phi}}\phi(x_1)\dots\phi(x_n)\Big|_{\phi=0}. \tag{1}$$

Use this expression to prove Wick's theorem, i.e. show that

$$\langle \phi(x_1)\dots\phi(x_{2n})\rangle_0 = D_F(x_1-x_2)D_F(x_3-x_4)\dots D_F(x_{2n-1}-x_{2n}) + \text{'all other contractions'}.$$

Hint: First show that

$$\langle \phi(x_1)\dots\phi(x_{2n})\rangle_0 = \frac{1}{2^n n!}\sum_\sigma D_F(x_{\sigma_1}-x_{\sigma_2})D_F(x_{\sigma_3}-x_{\sigma_4})\dots D_F(x_{\sigma_{2n-1}}-x_{\sigma_{2n}}), \tag{2}$$

where the sum is over all permutations σ of $2n$ elements, and then bring this into the required form.

$$\langle \phi(x_1)\dots\phi(x_{2n})\rangle_0 := e^{\frac{1}{2}\frac{\delta}{\delta\phi}D_F\frac{\delta}{\delta\phi}}\phi(x_1)\dots\phi(x_{2n})\Big|_{\phi=0} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2}\frac{\delta}{\delta\phi}D_F\frac{\delta}{\delta\phi}\right)^k\prod_{i=1}^{2n}\phi(x_i)\Big|_{\phi=0}$$

$$=\sum_{k=0}^{n}\frac{1}{2^k k!}\left(\frac{\delta}{\delta\phi}D_F\frac{\delta}{\delta\phi}\right)^k\prod_{i=1}^{2n}\phi(x_i)\Big|_{\phi=0} = \frac{1}{2^n n!}\left(\frac{\delta}{\delta\phi}D_F\frac{\delta}{\delta\phi}\right)^n\prod_{i=1}^{2n}\phi(x_i),$$

where all terms with $k>n$ vanish since they contain a larger number of functional derivatives $\frac{\delta}{\delta\phi}$ than fields ϕ. On the other hand, all terms with $k<n$ contain more fields than functional derivatives so that, regardless of the prefactors containing powers of D_F, they too will vanish when setting ϕ to zero.
Only $k=n$ survives for which we insert the functional inner product given by

$$\frac{\delta}{\delta\phi}D_F\frac{\delta}{\delta\phi} := \int dy_1\,dy_2\,\frac{\delta}{\delta\phi(y_1)}D_F(y_1-y_2)\frac{\delta}{\delta\phi(y_2)}$$

to get

$$\langle \phi(x_1)\dots\phi(x_{2n})\rangle_0 = \frac{1}{2^n n!}\prod_{j=1}^{n}\int dy_j\,dy_{j+n}\,\frac{\delta}{\delta\phi(y_j)}D_F(y_j-y_{j+n})\frac{\delta}{\delta\phi(y_{j+n})}\underbrace{\prod_{i=1}^{2n}\phi(x_i)}_{\prod_{j=1}^{n}\phi(x_j)\phi(x_{j+n})}.$$

Now we let the $2n$ functional derivatives act on the $2n$ fields. During the process, we are careful to obey the product rule which gives a sum over all permutations of field and derivative coordinates such that

$$\langle \phi(x_1) \dots \phi(x_{2n}) \rangle_0 = \frac{1}{2^n n!} \prod_{j=1}^{n} \sum_{\sigma} \int dy_j \, dy_{j+n} \underbrace{\frac{\delta \phi(x_{\sigma_j})}{\delta \phi(y_{\sigma_j})}}_{\delta(x_{\sigma_j} - y_{\sigma_j})} D_F(y_{\sigma_j} - y_{\sigma_{j+n}}) \underbrace{\frac{\delta \phi(x_{\sigma_{j+n}})}{\delta \phi(y_{\sigma_{j+n}})}}_{\delta(x_{\sigma_{j+n}} - y_{\sigma_{j+n}})}$$

$$= \frac{1}{2^n n!} \prod_{j=1}^{n} \sum_{\sigma} D_F(x_{\sigma_j} - x_{\sigma_{j+n}}) = \frac{1}{2^n n!} \sum_{\sigma} \prod_{j=1}^{n} D_F(x_{\sigma_j} - x_{\sigma_{j+n}})$$

The sum over permutations still contains a lot of redundancy. Not only is every distinct possibility of contracting the fields represented 2^n-times in this sum because $D_F(x_i - x_j) = D_F(x_j - x_i)$, but also $n!$-times because every possible order in which the Feynman propagators could appear is also accounted for. Since

$$D_F(x_1 - x_2) \dots D_F(x_{i_1} - x_{j_1}) \dots D_F(x_{i_2} - x_{j_2}) \dots D_F(x_{2n-1} - x_{2n}) = D_F(x_1 - x_2) \dots D_F(x_{i_2} - x_{j_2}) \dots D_F(x_{i_1} - x_{j_1}) \dots D_F(x_{2n-1} - x_{2n}),$$

we can get rid of this redundancy by letting it cancel with the prefactor $\frac{1}{2^n n!}$

$$\langle \phi(x_1) \dots \phi(x_{2n}) \rangle_0 = \sum_{\substack{\text{distinct} \\ \text{contractions}}} \prod_{j=1}^{n} D_F(x_{\sigma_j} - x_{\sigma_{j+n}})$$

$$= D_F(x_1 - x_2) \, D_F(x_3 - x_4) \dots D_F(x_{2n-1} - x_{2n}) + \text{'all other contractions'}$$

Problem 3.2 (Two-point function and functional determinant for a free theory)

In this exercise we will evaluate the two-point function

$$\langle\Omega|T\,\phi(x_1)\,\phi(x_2)|\Omega\rangle = \lim_{T\to\infty} \frac{\int \mathcal{D}\phi\,\phi(x_1)\,\phi(x_2)\,e^{iS[\phi]}}{\int \mathcal{D}\phi\, e^{iS[\phi]}} \tag{3}$$

for the free real scalar field theory,

$$S[\phi] = \int d^4x\,\mathcal{L} = \int d^4x\,\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - (m^2-i\epsilon)\,\phi^2\right), \tag{4}$$

by means of path integral techniques. To evaluate the path integral, we first introduce a UV- and IR- cutoff, i.e. we discretize space and restrict it to a finite volume. Therefore, we get

$$\mathcal{D}\phi = \prod_i d\phi(x_i), \qquad \text{(prefactors omitted since we calculate ratios in eq. (3))}$$

$$\phi(x_i) = \frac{1}{V}\sum_n e^{-ik_n\cdot x_i}\,\hat{\phi}(k_n), \qquad \text{(sum over } n \text{ finite due to UV-cutoff)}$$

where $k_n^\mu = \frac{2\pi n^\mu}{L}$ with $n^\mu\in\mathbb{Z}$, $|k^\mu|<\frac{\pi}{a}$ and $V=L^4$. L and a are the order parameters of volume and lattice spacing, respectively.

a) Why is the measure, when we go to Fourier space, given by

$$\mathcal{D}\phi(x) = J\prod_{k_n^0>0} d\,\mathrm{Re}\,\hat{\phi}(k_n)\; d\,\mathrm{Im}\,\hat{\phi}(k_n), \tag{5}$$

with some constant J, and not by (5) with k_n^0 unconstrained?

Since $\phi(x)\in\mathbb{R}\ \forall x$ is real, we have $\hat{\phi}(-k_n) = \hat{\phi}^*(k_n)$. Therefore, $\mathcal{D}\phi(x) = J\prod_{k_n^0>0} d\mathrm{Re}\,\hat{\phi}(k_n)\,d\mathrm{Im}\,\hat{\phi}(k_n)$ already integrates over all d.o.f. Leaving k_n^0 unconstrained amounts to treating a complex field.

b) By proceeding as in problem 2.2 d) and e), show that

$$\int \mathcal{D}\phi\, e^{iS[\phi]} = J\prod_{k_n^0>0}\sqrt{\frac{-i\pi V}{m^2-i\epsilon-k_n^2}}\sqrt{\frac{-i\pi V}{m^2-i\epsilon-k_n^2}} = J\prod_{\text{all }k_n}\sqrt{\frac{-i\pi V}{m^2-i\epsilon-k_n^2}} \tag{6}$$

Hint: Use (or show) that for $\mathrm{Re}(b)>0$, the Gaussian integral $\int_{-\infty}^{\infty}dx\,e^{-bx^2}$ is well-defined and gives $\sqrt{\frac{\pi}{b}}$. Argue why this theorem is applicable here.

We start by writing the action given in eq. (4) in a discretised form and transforming it to Fourier space

$$S[\phi] = \int d^4x\, \mathcal{L} = \int d^4x\, \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - (m^2-i\epsilon)\phi^2\right) \longrightarrow \sum_i \frac{1}{2}\left(\frac{\partial}{\partial x^i}\phi(x_i)\frac{\partial}{\partial x_i}\phi(x_i) - (m^2-i\epsilon)\phi^2(x_i)\right)$$

$$S[\hat\phi] = \sum_i \frac{1}{2}\left(\frac{\partial}{\partial x^i}\frac{1}{V}\sum_n e^{-ik_n\cdot x_i}\hat\phi(k_n)\,\frac{\partial}{\partial x_i}\frac{1}{V}\sum_m e^{-ik_m\cdot x_i}\hat\phi(k_m) - (m^2-i\epsilon)\frac{1}{V^2}\sum_{n,m} e^{-i(k_n+k_m)\cdot x_i}\hat\phi(k_n)\hat\phi(k_m)\right)$$

$$= \frac{1}{2V^2}\sum_{n,m}(-ik_n)(-ik_m)\hat\phi(k_n)\hat\phi(k_m)\underbrace{\sum_i e^{-i(k_n+k_m)\cdot x_i}}_{V\delta_{k_n,-k_m}} - (m^2-i\epsilon)\hat\phi(k_n)\hat\phi(k_m)\underbrace{\sum_i e^{-i(k_n+k_m)\cdot x_i}}_{V\delta_{k_n,-k_m}}$$

$$= \frac{1}{2V}\sum_n (-ik_n)ik_n\,\hat\phi(k_n)\underbrace{\hat\phi(-k_n)}_{\hat\phi^\dagger(k_n)} - (m^2-i\epsilon)\,\hat\phi(k_n)\underbrace{\hat\phi(-k_n)}_{\hat\phi^\dagger(k_n)} = \frac{1}{2V}\sum_n \left(k_n^2-(m^2-i\epsilon)\right)|\hat\phi(k_n)|^2,$$

where $\hat\phi(-k_n)=\hat\phi^\dagger(k_n)$ ensures that $\phi(x_i)=\frac{1}{V}\sum_n e^{-ik_n\cdot x}\hat\phi(k_n)$ is self-adjoint, i.e.

$$\phi^\dagger(x_i) = \frac{1}{V}\sum_n e^{+ik_n\cdot x}\hat\phi^\dagger(k_n) \overset{n\to-n}{=} \frac{1}{V}\sum_{-n} e^{-ik_n\cdot x_i}\underbrace{\hat\phi^\dagger(-k_n)}_{\hat\phi(k_n)} = \phi(x_i), \quad \text{with } k^\mu_{-n} = \frac{2\pi(-n^\mu)}{L} = -k^\mu_n$$

We can now insert the Fourier space action into the path integral, writing $|\hat\phi(k_n)|^2$ as $\mathrm{Re}^2\hat\phi(k_n)+\mathrm{Im}^2\hat\phi(k_n)$, and using the Gaussian integral to solve

$$\int D\phi\, e^{iS[\phi]} = \int J\prod_{k_n^0>0} d\mathrm{Re}\,\hat\phi(k_n)\, d\mathrm{Im}\,\hat\phi(k_n)\; e^{\frac{i}{2V}\underbrace{\sum_n (k_n^2-(m^2-i\epsilon))}_{\sum_{n>0}(k_n^2-(m^2-i\epsilon))+(\underbrace{k_{-n}^2}_{k_n^2}-(m^2-i\epsilon)) = 2\sum_{n>0}(k_n^2-(m^2-i\epsilon))}\left(\mathrm{Re}^2\hat\phi(k_n)+\mathrm{Im}^2\hat\phi(k_n)\right)}$$

$$= J\prod_{k_n^0>0}\int d\mathrm{Re}\,\hat\phi(k_n)\, e^{-\frac{i}{V}(m^2-i\epsilon-k_n^2)\mathrm{Re}^2\hat\phi(k_n)}\int d\mathrm{Im}\,\hat\phi(k_n)\, e^{-\frac{i}{V}(m^2-i\epsilon-k_n^2)\mathrm{Im}^2\hat\phi(k_n)}$$

$$= J\prod_{k_n^0>0}\sqrt{\frac{\pi}{\frac{i}{V}(m^2-i\epsilon-k_n^2)}}\;\underbrace{\sqrt{\frac{\pi}{\frac{i}{V}(m^2-i\epsilon-k_n^2)}}}_{k_n\to-k_n} = J\prod_{\text{all } k_n}\sqrt{\frac{-i\pi V}{m^2-i\epsilon-k_n^2}}$$

We used the Gaussian integral in the second to last step with $b=\frac{i}{V}(m^2-i\epsilon-k_n^2)$. As to its applicability, we note that since $\epsilon>0$ is a positive constant, we have that $\mathrm{Re}(b) = \mathrm{Re}\left(\frac{1}{V}(\epsilon+i(m^2-k_n^2))\right) = \frac{\epsilon}{V}>0$.

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c) Before we continue with the two-point function, we want to relate eq. (6) with the functional determinant. To do so, convince yourself that

$$\left(\prod_k \int d\xi_k\right) e^{-\xi_i B^{ij}\xi_j} = \prod_i \sqrt{\frac{\pi}{b^i}} = \frac{\text{const}}{\sqrt{\det B}} \qquad (7)$$

for some symmetric positive definite $N \times N$-matrix (more generally, it suffices that the eigenvalues have a positive real part) and $\vec{\xi} \in \mathbb{R}^N$.

Hint: Do an integral transformation to go to the diagonal space of B.

Every real symmetric matrix B can be diagonalized in the form

$$B = Q D Q^T,$$

where D is diagonal and Q is an orthognal matrix satisfying $Q^T = Q^{-1}$ and hence

$$1 = \det(\mathbb{1}_N) = \det(Q^T Q) = \det(Q^T)\det(Q) = \det{}^2(Q) \implies \det(Q) = \pm 1.$$

Following the hint, we now transform the integral

$$\left(\prod_{k=1}^N \int d\xi_k\right) e^{-\xi_i B^{ij}\xi_j} = \int d^N\vec{\xi}\, e^{-\vec{\xi}^T B \vec{\xi}} = \int d^N\vec{\xi}\, e^{-\vec{\xi}^T \underbrace{Q^T Q}_{\mathbb{1}_N} B \underbrace{Q^T Q}_{\mathbb{1}_N} \vec{\xi}} = \int d^N\vec{\xi}\, e^{-\underbrace{(Q\vec{\xi})^T}_{=:\vec{\zeta}^T} D \underbrace{(Q\vec{\xi})}_{=:\vec{\zeta}}}$$

$$= \int d^N\vec{\zeta}\, \underbrace{|\det(Q)|}_{1} e^{-\vec{\zeta}^T D \vec{\zeta}} = \left(\prod_{k=1}^N \int d\zeta_k\right) e^{-\zeta_i D^{ij}\zeta_j} = \left(\prod_{k=1}^N \int d\zeta_k\right) e^{-b^i \zeta_i^2}$$

where all the Eigenvalues b_i are real and positive, i.e. $\mathrm{Re}(b_i) > 0 \ \forall i$, so that we may again evaluate the above expression as a Gaussian integral

$$\left(\prod_{k=1}^N \int d\xi_k\right) e^{-\xi_i B^{ij}\xi_j} = \prod_{k=1}^N \int d\zeta_k\, e^{-b^k \zeta_k^2} = \prod_{k=1}^N \sqrt{\frac{\pi}{b^k}} = \frac{\text{const}}{\sqrt{\det B}}, \quad \text{where const} = \pi^{\frac{N}{2}}$$

d) Now rewrite the action in eq. (4) into the form

$$S[\phi] = \phi \cdot D \cdot \phi + \text{surface terms}$$

with D some differential operator and argue in analogy to c) that

$$\int \mathcal{D}\phi \, e^{iS[\phi]} = \frac{\text{const}}{\sqrt{\det D}}.$$

Further, give the differential operator D. Hint: You can neglect the surface terms in your argument because we are assuming natural boundary conditions.

The action $S[\phi]$ can be brought into the required form using partial integration

$$S[\phi] = \int d^4x \, \frac{1}{2}\left(\partial_\mu \phi(x) \partial^\mu \phi(x) - (m^2 - i\epsilon)\phi^2(x)\right) = \int d^4x \, \phi(x) \frac{1}{2}\left(-\partial_x^2 - (m^2 - i\epsilon)\right)\phi(x) + \frac{1}{2} e_\mu \phi(x) \partial^\mu \phi(x)\Big|_{\text{surface}}$$

$$= i\int d^4x \int d^4y \, \phi(x) \frac{i}{2}\left(\partial_y^2 + (m^2 - i\epsilon)\right)\delta(x-y)\phi(y) + \frac{1}{2} e_\mu \phi(x) \partial^\mu \phi(x)\Big|_{\text{surface}}$$

$$= i\phi \cdot D \cdot \phi + \text{surface terms}, \quad \text{with } D = \frac{i}{2}\left(\partial_y^2 + (m^2 - i\epsilon)\right)\delta(x-y) = D(x-y)$$

$$\int \mathcal{D}\phi \, e^{iS[\phi]} \underset{\text{neglecting surface terms}}{=} \left(\int \prod_i d\phi(x_i)\right) e^{-\phi \cdot D \cdot \phi} \underset{\text{using eq. (7)}}{=} \frac{\text{const}}{\sqrt{\det D}}$$

e) After this short interlude, we want to calculate the numerator on the right hand side of eq. (3). Therefore, use the same regularisation as for the denominator and go again to Fourier space. Then integrate out the Fourier coefficients. You should find the following expression for the numerator:

$$\int \mathcal{D}\phi \, \phi(x_1) \phi(x_2) \, e^{iS[\phi]} = \frac{J}{V^2} \sum_m e^{-ik_m \cdot (x_1 - x_2)} \frac{-iV}{m^2 - i\epsilon - k_m^2} \left(\prod_{k_n^0 > 0} \frac{-i\pi V}{m^2 - i\epsilon - k_n^2}\right). \qquad (8)$$

Now take the ratio of eqs. (8) and (6) and explain why it should actually be

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x_1 - x_2)}.$$

Hint: To evaluate the path integral, you will need also the higher momenta of the Gaussian integral.

We now leave the path laid out by the exercise and pursue a shorter and more elegant way of proving eq. (8) by introducing a physically meaningless but useful source $J(x)$. Doing so allows us to reformulate the path integral at hand as

$$I := \int \mathcal{D}\phi\, \phi(x_1)\, \phi(x_2)\, e^{iS[\phi]} = \int \mathcal{D}\phi\, \phi(x_1)\, \phi(x_2)\, e^{iS[\phi] + J\cdot\phi}\Big|_{J=0}$$

$$= \int \mathcal{D}\phi\, \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)}\, e^{-\phi\cdot D\cdot\phi + J\cdot\phi}\Big|_{J=0} = \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \int \mathcal{D}\phi\, e^{-\phi\cdot D\cdot\phi + J\cdot\phi}\Big|_{J=0}$$

The exponent can now be rewritten by completing the square

$$-\phi\cdot D\cdot\phi + J\cdot\phi = -\underbrace{\left(\phi - \tfrac{1}{2}D^{-1}\cdot J\right)}_{\phi'}\cdot D\cdot\underbrace{\left(\phi - \tfrac{1}{2}D^{-1}\cdot J\right)}_{\phi'} + \tfrac{1}{4} J\cdot D^{-1}\cdot J$$

We can thus transform the integral from $\phi \rightarrow \phi' = \phi - \frac{1}{2}D^{-1}\cdot J$ and $\mathcal{D}\phi \longrightarrow \mathcal{D}\phi' = \mathcal{D}\phi$ to achieve the same Gaussian integral we calculated in part d) times an exponential of $J(x)$ unaffected by the path integration:

$$I = \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)}\, e^{\frac{1}{4}J\cdot D^{-1}\cdot J} \underbrace{\int \mathcal{D}'\phi\, e^{-\phi'\cdot D\cdot\phi'}}_{\sqrt{\frac{\pi^n}{\det D}}}\Big|_{J=0}$$

$$= \sqrt{\frac{\pi^n}{\det D}}\, \frac{\delta}{\delta J(x_2)}\, \frac{1}{4}\left(\int d^4x \int d^4y\, \underbrace{\frac{\delta J(x)}{\delta J(x_1)}}_{\delta(x-x_1)} D^{-1}(x-y)\, J(y) + J(x)\, D^{-1}(x-y) \underbrace{\frac{\delta J(y)}{\delta J(x_1)}}_{\delta(y-x_1)}\right) e^{\frac{1}{4}J\cdot D^{-1}\cdot J}\Big|_{J=0}$$

$$= \sqrt{\frac{\pi^n}{\det D}}\, \frac{1}{4}\left(\left(\int d^4y\, D^{-1}(x_1-y) \underbrace{\frac{\delta J(y)}{\delta J(x_2)}}_{\delta(y-x_2)} + \int d^4x\, \underbrace{\frac{\delta J(x)}{\delta J(x_2)}}_{\delta(x-x_2)} D^{-1}(x-x_1)\right) e^{\frac{1}{4}J\cdot D^{-1}\cdot J} + \begin{array}{l}\text{second term}\\ \text{from product}\\ \text{rule that vanishes}\\ \text{when } J\to 0\end{array}\right)\Big|_{J=0}$$

$$= \sqrt{\frac{\pi^n}{\det D}}\, \frac{1}{4}\left(D^{-1}(x_1-x_2) + D^{-1}(x_2-x_1)\right),$$

where D^{-1} can be shown to be $2D_F$, D_F the Feynman propagator:

$$\int d^4y\, D(x-y)\, 2D_F(y-z) = \int d^4y\, \frac{i}{2}\left(\partial_y^2 + (m^2 - i\epsilon)\right)\delta(x-y)\, 2\int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(y-z)}}{p^2 - m^2 + i\epsilon}$$

$$= -\int \frac{d^4p}{(2\pi)^4} \left(\partial_x^2 + (m^2 - \epsilon)\right) \frac{e^{-ip(x-z)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} \left(p^2 - (m^2 - i\epsilon)\right) \frac{e^{-ip(x-z)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-z)} = \delta^{(4)}(x-z)$$

Taking into account that $D_F(x_1-x_2) = D_F(x_2-x_1)$, we may simplify I.

$$I = \sqrt{\frac{\pi^n}{\det D}}\,\frac{1}{4}\left(2D_F(x_1-x_2) + 2D_F(x_2-x_1)\right) = \sqrt{\frac{\pi^n}{\det D}}\;D_F(x_1-x_2)$$

Now taking the ratio of our results of part e) and d), we get the following expression for the two-point function of the free real scalar field,

$$\langle\Omega|T\,\phi(x_1)\,\phi(x_2)|\Omega\rangle = \lim_{T\to\infty}\frac{\int D\phi\,\phi(x_1)\,\phi(x_2)\,e^{iS[\phi]}}{\int D\phi\,e^{iS[\phi]}} = \frac{\sqrt{\frac{\pi^n}{\det D}}\;D_F(x_1-x_2)}{\sqrt{\frac{\pi^n}{\det D}}} = D_F(x_1-x_2).$$

Remark: Unfortunately, even though being shorter and more elegant our way of introducing a source J to arrive at the above result completely evades the point of this exercise, which is to show that the result also holds when restricting and discretizing space.