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Use a pen!

General Relativity - Exercise Sheet 4

Problem 1 (Straight back to the sphere) [10 points]

Last time, you derived the metric for a sphere in $\mathbb{R}^3$,

$$\mathrm{diag}(\{g_{ij}\}) = (1, r^2, r^2\sin^2\theta),$$

and limited yourself to the surface. This time, we will not do so and leave r free.

a) Calculate the Christoffel symbols Γ^i_{jk} of this metric.

Surpassing the instructions of exercise 2. b) on sheet 3, we already calculated all 27 space Christoffel symbols, all the while leaving $x^1 = r$ unconstrained. We found a total of nine symbols different from zero. These were

$$\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r\sin^2\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta.$$

b) The geodesic equations read

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0.$$

Do they describe straight lines? Should this depend on the coordinate system we're using?

Writing the geodesic equations explicitly, we get

$$\ddot{r} + (-r)\dot{\theta}\dot{\theta} + (-r\sin^2\theta)\dot{\phi}\dot{\phi} = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi} = 0,$$

$$\ddot{\theta} + \frac{1}{r}\dot{r}\dot{\theta} + \frac{1}{r}\dot{\theta}\dot{r} + (-\sin\theta\cos\theta)\dot{\phi}\dot{\phi} = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0,$$

$$\ddot{\phi} + \frac{1}{r}\dot{r}\dot{\phi} + \frac{1}{r}\dot{\phi}\dot{r} + \cot\theta\,\dot{\theta}\dot{\phi} + \cot\theta\,\dot{\phi}\dot{\theta} = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0.$$

Solving for the second-order time-derivatives, the geodesic equations result in

$$\ddot{r} = r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2, \qquad \ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\,\dot{\phi}^2, \qquad \ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} - 2\cot\theta\,\dot{\theta}\dot{\phi}.$$

Since we started out deriving the metric $\mathrm{diag}(\{g_{ij}\}) = (1, r^2, r^2\sin^2\theta)$ from flat Minkowskian space, a property that remains unaffected by a transformation to spherical coordinates, the equations of motion above, obtained from the geodesic equation, describe motion in flat space. They should therefore, when solved, yield straight lines.
At least for radial motion, i.e. along the r-dimension, it is easy to see that this is true. Radial motion implies $\theta = c_1$, $\phi = c_2$, with c_1, c_2 constant so that the right hand side of all geodesic eqns. and the left hand side of the second and third vanish and we are left with $\ddot{r} = 0$, solved by a straight line $r(t) = v \cdot t + d$.
The outcome/behavior of no physical process should depend on the coordinate system chosen to describe it.

Problem 2 (Transforming the metric) [10 points]

If we transform the metric by a scalar factor $e^{2\varphi}$, with $\varphi = \varphi(x^\mu)$ a function of all coordinates, the metric becomes

$$g \longrightarrow g' = e^{2\varphi} g. \qquad (1)$$

a) By plugging in the definition into the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, show that they will transform as

$$\Gamma^\alpha_{\mu\nu} \longrightarrow \Gamma'^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \delta^\alpha_\mu \partial_\nu \varphi + \delta^\alpha_\nu \partial_\mu \varphi - g^{\alpha\beta} g_{\mu\nu} \partial_\beta \varphi \qquad (2)$$

The Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ can be expressed in terms of derivatives of the metric as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right).$$

05.05.2015

Keeping in mind that $g'_{\alpha\beta}\, g'^{\beta\gamma} = \delta_\alpha^{\gamma}$, i.e. $g'^{\beta\gamma} = e^{-2\varphi} g^{\beta\gamma}$, we insert the transformed metric to arrive at eqn. (2).

$$\Gamma'^{\alpha}_{\mu\nu} = \frac{1}{2} g'^{\alpha\beta}\left(\frac{\partial g'_{\beta\mu}}{\partial x^\nu} + \frac{\partial g'_{\beta\nu}}{\partial x^\mu} - \frac{\partial g'_{\mu\nu}}{\partial x^\beta}\right)$$

$$= \frac{1}{2} g^{\alpha\beta} e^{-2\varphi}\left(\frac{\partial}{\partial x^\nu}\left(g_{\beta\mu} e^{2\varphi}\right) + \frac{\partial}{\partial x^\mu}\left(g_{\beta\nu} e^{2\varphi}\right) - \frac{\partial}{\partial x^\beta}\left(g_{\mu\nu} e^{2\varphi}\right)\right) \checkmark$$

$$= \frac{1}{2} g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right) + \frac{1}{2} g^{\alpha\beta} e^{-2\varphi}\Big(g_{\beta\mu}\, 2e^{2\varphi}\frac{\partial\varphi}{\partial x^\nu} + g_{\beta\nu}\, 2e^{2\varphi}\frac{\partial\varphi}{\partial x^\mu}$$

$$- g_{\mu\nu}\, 2e^{2\varphi}\frac{\partial\varphi}{\partial x^\beta}\Big) = \Gamma^{\alpha}_{\mu\nu} + g^{\alpha\beta}\left(g_{\beta\mu}\,\partial_\nu\varphi + g_{\beta\nu}\,\partial_\mu\varphi - g_{\mu\nu}\,\partial_\beta\varphi\right) \checkmark$$

$$= \Gamma^{\alpha}_{\mu\nu} + \delta^{\alpha}_{\mu}\,\partial_\nu\varphi + \delta^{\alpha}_{\nu}\,\partial_\mu\varphi - g^{\alpha\beta} g_{\mu\nu}\,\partial_\beta\varphi \quad \checkmark$$

b) Null geodesics ($ds = 0$) have a tangent vector that is light-like,

$$\dot{x}^\mu \dot{x}_\mu = 0.$$

Show that null geodesics are invariant under the transformation in a), i.e. that null geodesics of the metric g will remain null geodesics of the metric g', if the curve parameter λ is transformed as

$$d\lambda \to d\lambda' = e^{2\varphi} d\lambda. \qquad (3)$$

$$\ddot{x}'^{\alpha} + \Gamma'^{\alpha}_{\mu\nu}\dot{x}'^{\mu}\dot{x}'^{\nu} := \frac{d}{d\lambda'}\frac{dx^\alpha}{d\lambda'} + \Gamma'^{\alpha}_{\mu\nu}\frac{dx^\mu}{d\lambda'}\frac{dx^\nu}{d\lambda'} = e^{-2\varphi}\frac{d}{d\lambda}\left(e^{-2\varphi}\frac{d}{d\lambda}x^\alpha\right)$$

$$+ e^{-4\varphi}\left(\Gamma^{\alpha}_{\mu\nu} + \delta^{\alpha}_{\mu}\partial_\nu\varphi + \delta^{\alpha}_{\nu}\partial_\mu\varphi - g^{\alpha\beta} g_{\mu\nu}\partial_\beta\varphi\right)\dot{x}^\mu\dot{x}^\nu$$

$$= e^{-4\varphi}\left(-2\frac{d\varphi}{d\lambda}\dot{x}^\alpha\right) + e^{-4\varphi}\ddot{x}^\alpha + e^{-4\varphi}\Big(\Gamma^{\alpha}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \partial_\nu\varphi\,\dot{x}^\nu\dot{x}^\alpha + \partial_\mu\varphi\,\dot{x}^\mu\dot{x}^\alpha - \partial^\alpha\varphi\,\underbrace{\dot{x}_\mu\dot{x}^\mu}_{0}\Big) \checkmark$$

$$= e^{-4\varphi}\left(\ddot{x}^\alpha + \Gamma^{\alpha}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - 2\frac{\partial\varphi}{\partial x^\mu}\frac{dx^\mu}{d\lambda}\dot{x}^\alpha + 2\partial_\mu\varphi\,\dot{x}^\mu\dot{x}^\alpha\right) = e^{-4\varphi}\left(\ddot{x}^\alpha + \Gamma^{\alpha}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) \checkmark$$

Therefore, we can say that

$$\text{null geodesics} \iff \ddot{x}^\alpha + \Gamma^{\alpha}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \iff \ddot{x}'^{\alpha} + \Gamma'^{\alpha}_{\mu\nu}\dot{x}'^{\mu}\dot{x}'^{\nu} = 0 \quad \checkmark$$

Problem 3 (An interesting line element) [20 points]

Consider the following line element,

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right].$$

a) Calculate the non-vanishing Christoffel symbols.

$$\{g_{\mu\nu}(t)\} = \begin{pmatrix} 1 & & & \\ & -a^2(t)/(1-kr^2) & & \\ & & -a^2(t)r^2 & \\ & & & -a^2(t)r^2\sin^2\theta \end{pmatrix}, \quad x = \begin{pmatrix} ct \\ r \\ \theta \\ \phi \end{pmatrix} \checkmark$$

$$\{g^{\mu\nu}(t)\} = \begin{pmatrix} 1 & & & \\ & -(1-kr^2)/a^2(t) & & \\ & & -1/(a^2(t)r^2) & \\ & & & -1/(a^2(t)r^2\sin^2\theta) \end{pmatrix}$$

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\frac{\partial g_{\nu\beta}}{\partial x^{\mu}} + \frac{\partial g_{\mu\beta}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right)$$

Since $g_{\mu\nu}$ is diagonal, we may perform a case analysis of the above definition of the Christoffel symbols to reduce the number of required calculations to find them all

if $\alpha \neq \mu \neq \nu$: $\quad \Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} \underbrace{g^{\alpha\beta}}_{\substack{\text{only} \neq 0 \\ \text{if } \alpha=\beta}}\Bigg(\underbrace{\frac{\partial g_{\nu\beta}}{\partial x^{\mu}}}_{\substack{\text{only} \neq 0 \\ \text{if } \nu=\beta}} + \underbrace{\frac{\partial g_{\mu\beta}}{\partial x^{\nu}}}_{\substack{\text{only} \neq 0 \\ \text{if } \mu=\beta}} - \underbrace{\frac{\partial g_{\mu\nu}}{\partial x^{\beta}}}_{\substack{\text{always } 0, \\ \text{since } \mu\neq\nu}}\Bigg) = 0$

if $\alpha \neq \mu = \nu$: $\quad \Gamma^{\alpha}{}_{\mu\mu} = \frac{1}{2} g^{\alpha\alpha}\Bigg(2\underbrace{\frac{\partial g_{\mu\alpha}}{\partial x^{\mu}}}_{0} - \frac{\partial g_{\mu\mu}}{\partial x^{\alpha}}\Bigg) = -\frac{1}{2} g^{\alpha\alpha}\frac{\partial g_{\mu\mu}}{\partial x^{\alpha}}$

if $\alpha = \mu \neq \nu$: $\quad \Gamma^{\alpha}{}_{\alpha\nu} = \frac{1}{2} g^{\alpha\alpha}\Bigg(\underbrace{\frac{\partial g_{\nu\alpha}}{\partial x^{\alpha}}}_{0} + \frac{\partial g_{\alpha\alpha}}{\partial x^{\nu}} - \underbrace{\frac{\partial g_{\alpha\nu}}{\partial x^{\alpha}}}_{0}\Bigg) = \frac{1}{2} g^{\alpha\alpha}\frac{\partial g_{\alpha\alpha}}{\partial x^{\nu}}$

if $\alpha = \mu = \nu$: $\quad \Gamma^{\alpha}{}_{\alpha\alpha} = \frac{1}{2} g^{\alpha\alpha}\left(2\frac{\partial g_{\alpha\alpha}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\alpha}}{\partial x^{\alpha}}\right) = \frac{1}{2} g^{\alpha\alpha}\frac{\partial g_{\alpha\alpha}}{\partial x^{\alpha}}$

From the first case, we now know that we only need to consider Christoffel symbols with at least one reoccuring index, thereby removing $\frac{4!}{(4-3)!} = 24$ entries from our to-do list previously containing $4^3 = 64$ items. The symmetry in μ, ν helps further.

05.05.2015

$\Gamma^0_{00} = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0} = \frac{1}{2}\, 1\, \frac{\partial 1}{\partial ct} = 0$, $\quad \Gamma^1_{11} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^1} = \frac{1}{2} \frac{1-kr^2}{a^2(t)} \frac{\partial}{\partial r} \frac{a^2(t)}{1-kr^2} = \frac{kr}{1-kr^2}$ ✓

$\Gamma^2_{22} = \frac{1}{2} g^{22} \underbrace{\frac{\partial g_{22}}{\partial x^2}}_{0} = 0$, $\quad \Gamma^3_{33} = \frac{1}{2} g^{33} \underbrace{\frac{\partial g_{33}}{\partial x^3}}_{0} = 0$

$\Gamma^0_{01} = \Gamma^0_{10} = \Gamma^0_{20} = \Gamma^0_{02} = \Gamma^0_{30} = \Gamma^0_{03} = 0$, because $g^{00} = 1$ constant

$\Gamma^1_{10} = \Gamma^1_{01} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^0} = \frac{1}{2} \frac{1-kr^2}{a^2(t)} \frac{\partial}{c\partial t} \frac{a^2(t)}{1-kr^2} = \frac{\dot{a}(t)}{c\,a(t)}$ ✓

$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^1_{13} = \Gamma^1_{31} = 0$, because $g_{11} = -\frac{a^2(t)}{1-kr^2}$ does not depend on θ, ϕ

$\Gamma^2_{02} = \Gamma^2_{20} = \frac{1}{2} \frac{1}{a^2(t) r^2} \frac{\partial}{c\partial t} a^2(t) r^2 = \frac{\dot{a}(t)}{c\,a(t)}$ ✓

$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} \frac{1}{a^2(t) r^2} \frac{\partial}{\partial r} a^2(t) r^2 = \frac{1}{r}$, ✓ $\quad \Gamma^2_{32} = \Gamma^2_{23} = 0$, because $\frac{\partial}{\partial \phi} a^2(t) r^2 = 0$

$\Gamma^3_{30} = \Gamma^3_{03} = \frac{\dot{a}(t)}{c\,a(t)}$ ✓, $\quad \Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r}$ ✓, $\quad \Gamma^3_{32} = \Gamma^3_{23} = \frac{1}{2} \frac{1}{a^2(t) r^2 \sin^2\theta} \frac{\partial}{\partial \theta} a^2(t) r^2 \sin^2\theta = \cot\theta$ ✓

$\Gamma^0_{11} = -\frac{1}{2} g^{00} \frac{\partial g_{11}}{\partial x^0} = \frac{1}{2} \frac{\partial}{c\partial t} \frac{a^2(t)}{1-kr^2} = \frac{a(t)\dot{a}(t)}{c(1-kr^2)}$ ✓, $\quad \Gamma^0_{22} = \frac{1}{2} \frac{\partial}{c\partial t} a^2(t) r^2 = \frac{r^2}{c} a(t) \dot{a}(t)$ ✓

$\Gamma^0_{33} = \frac{r^2}{c} a(t) \dot{a}(t) \sin^2\theta$ ✓

$\Gamma^1_{00} = -\frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial x^1} = 0$, $\quad \Gamma^1_{22} = -\frac{1}{2} \frac{1-kr^2}{a^2(t)} \frac{\partial}{\partial r} a^2(t) r^2 = -r(1-kr^2)$, ✓ $\quad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta$ ✓

$\Gamma^2_{00} = 0$, $\quad \Gamma^2_{11} = 0$, $\quad \Gamma^2_{33} = -\frac{1}{2} \frac{1}{a^2(t) r^2} \frac{\partial}{\partial \theta} a^2(t) r^2 \sin^2\theta = -\sin\theta\cos\theta$ ✓

$\Gamma^3_{00} = 0$, $\quad \Gamma^3_{11} = 0$, $\quad \Gamma^3_{22} = 0$, because the metric does not depend on ϕ

b) Write down what geodesic a particle follows that only moves radially, i.e. in x^1-direction?

For such a particle, we have θ and ϕ constant. Only two geodesic eqs. remain.

$$x^{0''} + \Gamma^0_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = c t'' + \frac{a(t)\dot{a}(t)}{c(1-kr^2)} r'^2 = 0 \quad ✓$$

$$x^{1''} + \Gamma^1_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = r'' + \frac{kr}{1-kr^2} r'^2 + 2 \frac{\dot{a}(t)}{c\,a(t)} r' t' = 0 \quad ✓$$

where a prime marks the derivative w.r.t. τ.

c) What is the radial distance to a light source ($ds = 0$),

$$r = \int_{t_e}^{t_{obs}} dr',$$

in terms of dt, if $k = 0$ (we will see what this means in the future).
Given $\frac{dr}{dt_e} = 0$, show that $\frac{dt_{obs}}{dt_e} = \frac{a(t_{obs})}{a(t_e)}$. With $dt = \frac{1}{f}$, find a relation between wavelength $\lambda = \frac{c}{f}$ and $a(t_{obs})/a(t_e)$. Rewrite it in terms of the parameter z with $z = \frac{\lambda_{obs} - \lambda_e}{\lambda_e}$.
Cosmologists call z redshift and $a(t)$ expansion factor. What redshift do you find for signals sent out when the Universe was only half as large as today?

For $k = 0$, the line element reads

$$ds^2 = c^2 dt^2 - a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)\right]. \checkmark$$

Since we are only interested in the radial distance to the light source, we keep θ and ϕ constant, i.e. $d\theta = d\phi = 0$. Since $ds^2 = 0$ for light, we get

$$0 \overset{!}{=} ds^2 = c^2 dt^2 - a^2(t)\, dr^2 \implies dr = \frac{c\,dt}{a(t)} \quad \checkmark$$

$$\implies r = \int_{t_e}^{t_{obs}} dr' = c\int_{t_e}^{t_{obs}} \frac{dt}{a(t)}, \checkmark \qquad 0 \overset{!}{=} \frac{dr}{dt_e} = c\frac{d}{dt_e}\int_{t_e}^{t_{obs}} \frac{dt}{a(t)} \overset{\text{chain rule}}{=} c\frac{1}{a(t_{obs})}\frac{dt_{obs}}{dt_e} - c\frac{1}{a(t_e)}\frac{dt_e}{dt_e} \checkmark$$

Therefore, $\frac{dt_{obs}}{dt_e} = \frac{a(t_{obs})}{a(t_e)}$. Inserting $dt = \frac{1}{f} = \frac{1}{c/\lambda} = \frac{\lambda}{c}$ gives

$$\frac{\lambda_{obs}}{\lambda_e} = \frac{a(t_{obs})}{a(t_e)} \quad \text{and hence} \quad z = \frac{\lambda_{obs} - \lambda_e}{\lambda_e} = \frac{a(t_{obs}) - a(t_e)}{a(t_e)}. \checkmark$$

For $a(t_{obs}) = 2a(t_e)$, i.e. a signal sent out at a time when the universe was half of its size at the time of observance, we get a redshift of $z = 1$. ✓

07.05.2015

Problem 4 (Extra question: Equivalence)

If there were no space exploration and you didn't have means to travel far, how could you distinguish Earth being a sphere with 'regular' gravity from Earth as a flat disk accelerated upwards with g by a giant turtle?

Ignoring the fact that a sufficiently large turtle capable of accelerating the entire world might be able to produce what we have come to call Earth's gravity with her own mass, we could distinguish between the above cases by observing other objects in our solar system for extended periods of time. If they don't fall away or come rushing at us, we can't be continuously accelerating in one direction relative to them. ✓