String Theory

Solution to Assignment 4

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1 The quantum Virasoro algebra

In this exercise, we will show that in the quantized bosonic string theory the normal ordered Virasoro generators

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathcal{N}(\boldsymbol{\alpha}_{m-n} \cdot \boldsymbol{\alpha}_n) \tag{1}$$

satisfy the Virasoro algebra with central charge[a]

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12} m(m^2-1) \delta_{m,-n}. \tag{2}$$

In order to become more familiar with the normal ordering prescription, we will do this by brute force methods, i.e., by simply using the definition of the normal ordered generators L_m and then calculating their commutators. We will proceed in several smaller steps.

a) Explain why the normal ordering in L_m only affects L_0 and why the Virasoro generators L_m can be written in the following form,

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_{m-n} + \frac{1}{2} \sum_{n=1}^{\infty} \boldsymbol{\alpha}_{m-n} \cdot \boldsymbol{\alpha}_n. \tag{3}$$

b) Using $[X, YZ] = [X, Y]Z + Y[X, Z]$ and $[\alpha_m^\mu, \alpha_n^\nu] = m\, \eta^{\mu\nu} \delta_{m,-n}$, prove that for all $m, n \in \mathbb{Z}$,

$$[\alpha_m^\mu, L_n] = m \alpha_{m+n}^\mu. \tag{4}$$

c) Decompose the sum

$$\sum_{n=-\infty}^{\infty} = \sum_{n=-\infty}^{0} + \sum_{n=1}^{\infty} \tag{5}$$

as we did in eq. (3) to "solve" the normal ordering condition. Use the result of part b) to show that

$$\begin{aligned} [L_m, L_n] = &\frac{1}{2} \sum_{l=-\infty}^{0} [(m-l)\, \boldsymbol{\alpha}_l \cdot \boldsymbol{\alpha}_{m+n-l} + l\, \boldsymbol{\alpha}_{n+l} \cdot \boldsymbol{\alpha}_{m-l}] \\ &+ \frac{1}{2} \sum_{l=1}^{\infty} [(m-l)\, \boldsymbol{\alpha}_{m+n-l} \cdot \boldsymbol{\alpha}_l + l\, \boldsymbol{\alpha}_{m-l} \cdot \boldsymbol{\alpha}_{n+l}]. \end{aligned} \tag{6}$$

d) Make the substitution $p = n + l$ in the second and fourth term in eq. (6) and verify

$$
\begin{aligned}
[L_m, L_n] = {} & \frac{1}{2} \sum_{l=-\infty}^{0} (m-l)\, \boldsymbol{\alpha}_l \cdot \boldsymbol{\alpha}_{m+n-l} + \frac{1}{2} \sum_{p=-\infty}^{n} (p-n)\, \boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} \\
& + \frac{1}{2} \sum_{l=1}^{\infty} (m-l)\, \boldsymbol{\alpha}_{m+n-l} \cdot \boldsymbol{\alpha}_l + \frac{1}{2} \sum_{p=n+1}^{\infty} (p-n)\, \boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p .
\end{aligned} \tag{7}
$$

e) From now on, we will restrict ourselves to the case $n > 0$, as the other cases $n < 0$ and $n = 0$ are completely analogous. Show, therefore, that, for $n > 0$, the expression eq. (7) in part d) is equal to

$$
\begin{aligned}
[L_m, L_n] = {} & \frac{1}{2} \sum_{p=-\infty}^{0} (m-n)\, \boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} + \frac{1}{2} \sum_{p=1}^{n} (p-n)\, \boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} \\
& + \frac{1}{2} \sum_{p=n+1}^{\infty} (m-n)\, \boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p + \frac{1}{2} \sum_{p=1}^{n} (m-p)\, \boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p .
\end{aligned} \tag{8}
$$

Which of these terms are already normal-ordered?

f) Prove

$$
\sum_{p=1}^{n} (p-n)\, \boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} = \sum_{p=1}^{n} (p-n)\, \boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p + \sum_{p=1}^{n} (p-n)\, p D \delta_{m,-n}, \tag{9}
$$

and insert this for the second term in eq. (8) of part e).

g) Show that your result from part e) is now equivalent to

$$
[L_m, L_n] = \frac{1}{2} \sum_{l=-\infty}^{\infty} (m-n)\, \mathcal{N}(\boldsymbol{\alpha}_l \cdot \boldsymbol{\alpha}_{m+n-l}) + \frac{1}{2} D \sum_{l=1}^{n} (l^2 - nl) \delta_{m,-n}. \tag{10}
$$

h) Prove, e.g. by induction, the following identities,

$$
\sum_{q=1}^{n} q^2 = \frac{n}{6}(n+1)(2n+1), \tag{11}
$$

$$
\sum_{q=1}^{n} q = \frac{n}{2}(n+1), \tag{12}
$$

and use this to finally derive

$$
[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12} m(m^2-1) \delta_{m,-n}. \tag{13}
$$

from the expression in part g).

[a] A central charge, T_0, of a Lie algebra is a generator that commutes with all generators of the Lie algebra, $[T_a, T_0] = 0 \ \forall a$, but appears on the right hand side of some commutators, $[T_a, Tb] = c\, T_0 + \ldots$, for some T_a and T_b, with c being a constant. In the above Virasoro algebra, the role of $_0$ is played by the term proportional to $\delta_{m,-n}$, which should be viewed as an extra generator in addition to the L_m.

a) As stated in eq. (1), the quantum Virasoro generators are defined as

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathcal{N}(\boldsymbol{\alpha}_{m-n} \cdot \boldsymbol{\alpha}_n), \tag{14}$$

where the normal ordering operator acts as

$$\mathcal{N}(\alpha_m^\mu \, \alpha_n^\nu) = \begin{cases} \alpha_m^\mu \, \alpha_n^\nu & \text{for } m \leq n, \\ \alpha_n^\nu \, \alpha_m^\mu & \text{for } n < m, \end{cases} \tag{15}$$

and the components of the modes $\boldsymbol{\alpha}_m$, $m \in \mathbb{Z}$ satisfy the commutation relation

$$[\alpha_m^\mu, \alpha_n^\nu] = m \, \eta^{\mu\nu} \delta_{m,-n}, \tag{16}$$

i.e. the order of α_m^μ and α_n^ν only matters if $m = -n$. Looking at eq. (14), we see that this scenario can only arise if $m = 0$. For $m \neq 0$, $m - n$ can never equal $-n$. Therefore, we only need to worry about normal ordering when treating L_0.

For $m = 0$ and $n > 0$, the dot product in eq. (14) is already in normal order. For $m = 0$ and $n \leq 0$, the order is reversed. Since we just established that in all other L_m, $m \neq 0$, the order is arbitrary, we can suppress the normal ordering symbol altogether by rewriting

$$\begin{aligned} L_m &= \frac{1}{2} \sum_{n=-\infty}^{0} \mathcal{N}(\boldsymbol{\alpha}_{m-n} \cdot \boldsymbol{\alpha}_n) + \frac{1}{2} \sum_{n=1}^{\infty} \mathcal{N}(\boldsymbol{\alpha}_{m-n} \cdot \boldsymbol{\alpha}_n) \\ &= \frac{1}{2} \sum_{n=-\infty}^{0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_{m-n} + \frac{1}{2} \sum_{n=1}^{\infty} \boldsymbol{\alpha}_{m-n} \cdot \boldsymbol{\alpha}_n. \end{aligned} \tag{17}$$

b) We show by direct calculation that $[\alpha_m^\mu, L_n] = m \alpha_{m+n}^\mu \;\; \forall \, m, n \in \mathbb{Z}$.

$$\begin{aligned} [\alpha_m^\mu, L_n] &= \frac{1}{2} \sum_{l=-\infty}^{0} [\alpha_m^\mu, \alpha_{l,\nu} \, \alpha_{n-l}^\nu] + \frac{1}{2} \sum_{l=1}^{\infty} [\alpha_m^\mu, \alpha_{n-l}^\nu \, \alpha_{l,\nu}] \\ &= \frac{1}{2} \sum_{l=-\infty}^{0} \Big([\alpha_m^\mu, \alpha_{l,\nu}] \alpha_{n-l}^\nu + \alpha_{l,\nu} [\alpha_m^\mu, \alpha_{n-l}^\nu] \Big) + \frac{1}{2} \sum_{l=1}^{\infty} \Big([\alpha_m^\mu, \alpha_{n-l}^\nu] \alpha_{l,\nu} + \alpha_{n-l}^\nu [\alpha_m^\mu, \alpha_{l,\nu}] \Big) \\ &\overset{(16)}{=} \frac{m}{2} \sum_{l=-\infty}^{0} \Big(\eta^\mu{}_\nu \delta_{m,-l} \alpha_{n-l}^\nu + \eta^{\mu\nu} \delta_{m,l-n} \alpha_{l,\nu} \Big) + \frac{m}{2} \sum_{l=1}^{\infty} \Big(\eta^{\mu\nu} \delta_{m,l-n} \alpha_{l,\nu} + \eta^\mu{}_\nu \delta_{m,-l} \alpha_{n-l}^\nu \Big). \end{aligned} \tag{18}$$

We now carry out the above sums over l. When doing so, we have to bear in mind, however, that two of the four Kronecker deltas never contribute. Which ones depends on the values of m and n. E.g. take $m > n > 0$, then the first $\delta_{m,-l}$ contributes at $l = -m$ but $\delta_{m,l-n} = 1 \;\; \Leftrightarrow \;\; m + n = l$ is always zero since $m + n > 0$. In the second we sum over positive l, so this behavior is reversed. What we end up with is

$$[\alpha_m^\mu, L_n] = \frac{m}{2} \alpha_{m+n}^\mu + \frac{m}{2} \alpha_{m+n}^\mu = m \, \alpha_{m+n}^\mu. \tag{19}$$

c) Next we demonstrate eq. (6) $\forall\, m, n \in \mathbb{Z}$. Using $[XY, Z] = X[Y, Z] + [X, Z]Y$, we can write

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{l=-\infty}^{0} [\alpha_{l,\mu}\, \alpha^{\mu}_{m-l}, L_n] + \frac{1}{2}\sum_{l=1}^{\infty}[\alpha^{\mu}_{m-l}\, \alpha_{l,\mu}, L_n] \\
&= \frac{1}{2}\sum_{l=-\infty}^{0} \Big(\alpha_{l,\mu}[\alpha^{\mu}_{m-l}, L_n] + [\alpha_{l,\mu}, L_n]\alpha^{\mu}_{m-l}\Big) \\
&\quad + \frac{1}{2}\sum_{l=1}^{\infty}\Big(\alpha^{\mu}_{m-l}[\alpha_{l,\mu}, L_n] + [\alpha^{\mu}_{m-l}, L_n]\alpha_{l,\mu}\Big) \\
&\overset{(19)}{=} \frac{1}{2}\sum_{l=-\infty}^{0} \Big((m-l)\,\boldsymbol{\alpha}_l \cdot \boldsymbol{\alpha}_{m+n-l} + l\,\boldsymbol{\alpha}_{n+l} \cdot \boldsymbol{\alpha}_{m-l}\Big) \\
&\quad + \frac{1}{2}\sum_{l=1}^{\infty}\Big((m-l)\,\boldsymbol{\alpha}_{m+n-l} \cdot \boldsymbol{\alpha}_l + l\,\boldsymbol{\alpha}_{m-l} \cdot \boldsymbol{\alpha}_{n+l}\Big).
\end{aligned} \tag{20}$$

d) Replacing $p = n + l$ in terms two and four of eq. (20) evidently gives

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{l=-\infty}^{0} (m-l)\,\boldsymbol{\alpha}_l \cdot \boldsymbol{\alpha}_{m+n-l} + \frac{1}{2}\sum_{p=-\infty}^{n} (p-n)\,\boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} \\
&\quad + \frac{1}{2}\sum_{l=1}^{\infty}(m-l)\,\boldsymbol{\alpha}_{m+n-l} \cdot \boldsymbol{\alpha}_l + \frac{1}{2}\sum_{p=n+1}^{\infty} (p-n)\,\boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p.
\end{aligned} \tag{21}$$

e) Restricting to $n > 0$ and relabeling $l \to p$ in the first and third term, we may write $[L_m, L_n]$ as

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{p=-\infty}^{0} (m-p)\,\boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} + \frac{1}{2}\sum_{p=-\infty}^{n} (p-n)\,\boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} \\
&\quad + \frac{1}{2}\sum_{p=1}^{\infty}(m-p)\,\boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p + \frac{1}{2}\sum_{p=n+1}^{\infty} (p-n)\,\boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p.
\end{aligned} \tag{22}$$

We can now partially consolidate sums one and two and sums three and four by splitting sum two at $p = 0$ and sum three at $p = n$ (both operations are legal because $n > 0$) to get

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{p=-\infty}^{0} (m-n)\,\boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} + \frac{1}{2}\sum_{p=1}^{n}(p-n)\,\boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} \\
&\quad + \frac{1}{2}\sum_{p=n+1}^{\infty} (m-n)\,\boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p + \frac{1}{2}\sum_{p=1}^{n}(m-p)\,\boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p.
\end{aligned} \tag{23}$$

Assuming further $n > m > 0$, we check for normal ordering in each of the four sums:

sum	p-range	critical value	operator product	mode indices	normal ordered
1	$[-\infty, 0]$	$p = 0$	$\boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p}$	$0 < m + n$	✓
2	$[1, n]$	$p = n$	$\boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p}$	$n \not< m$	×
3	$[n+1, \infty]$	$p = n + 1$	$\boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p$	$m + 1 < n + 1$	✓
4	$[1, n]$	$p = 1$	$\boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p$	$m + n - 1 \not< 1$	×

f) By an application of the modes' commutation relation, we find

$$\begin{aligned}
\sum_{p=1}^{n}(p-n)\,\boldsymbol{\alpha}_p \cdot \boldsymbol{\alpha}_{m+n-p} &= \sum_{p=1}^{n}(p-n)\,\Big(\boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p + \underbrace{[\alpha_{p,\mu}, \alpha^{\mu}_{m+n-p}]}_{p\,\eta^{\mu}{}_{\mu}\delta_{p,p-m-n} = pD\delta_{m,-n}}\Big) \\
&= \sum_{p=1}^{n}(p-n)\,\boldsymbol{\alpha}_{m+n-p} \cdot \boldsymbol{\alpha}_p + \sum_{p=1}^{n}(p-n)\,pD\delta_{m,-n}.
\end{aligned} \tag{24}$$

g) By inserting eq. (24) for the second term in eq. (23), we get

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\sum_{p=-\infty}^{0}(m-n)\,\boldsymbol{\alpha}_p\cdot\boldsymbol{\alpha}_{m+n-p} + \frac{1}{2}\sum_{p=1}^{n}(p-n)\,\boldsymbol{\alpha}_{m+n-p}\cdot\boldsymbol{\alpha}_p + \frac{1}{2}\sum_{p=1}^{n}(p-n)\,pD\delta_{m,-n} \\
+ \frac{1}{2}\sum_{p=n+1}^{\infty}(m-n)\,\boldsymbol{\alpha}_{m+n-p}\cdot\boldsymbol{\alpha}_p + \frac{1}{2}\sum_{p=1}^{n}(m-p)\,\boldsymbol{\alpha}_{m+n-p}\cdot\boldsymbol{\alpha}_p
\end{aligned}
\tag{25}
$$

where sums two and five combine to give

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\sum_{p=-\infty}^{0}(m-n)\,\boldsymbol{\alpha}_p\cdot\boldsymbol{\alpha}_{m+n-p} + \frac{1}{2}\sum_{p=1}^{n}(m-n)\,\boldsymbol{\alpha}_{m+n-p}\cdot\boldsymbol{\alpha}_p \\
+ \frac{1}{2}\sum_{p=1}^{n}(p-n)\,pD\delta_{m,-n} + \frac{1}{2}\sum_{p=n+1}^{\infty}(m-n)\,\boldsymbol{\alpha}_{m+n-p}\cdot\boldsymbol{\alpha}_p.
\end{aligned}
\tag{26}
$$

By further joining sums two and four and reinserting the normal ordering operator, we obtain

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{p=-\infty}^{0}(m-n)\,\boldsymbol{\alpha}_p\cdot\boldsymbol{\alpha}_{m+n-p} + \frac{1}{2}\sum_{p=1}^{\infty}(m-n)\,\boldsymbol{\alpha}_{m+n-p}\cdot\boldsymbol{\alpha}_p + \frac{D}{2}\sum_{p=1}^{n}p(p-n)\delta_{m,-n} \\
&= \frac{1}{2}\sum_{p=-\infty}^{\infty}(m-n)\,\mathcal{N}(\boldsymbol{\alpha}_p\cdot\boldsymbol{\alpha}_{m+n-p}) + \frac{D}{2}\sum_{p=1}^{n}p(p-n)\,\delta_{m,-n}.
\end{aligned}
\tag{27}
$$

h) We use induction to proof the following two identities.

1. $\sum_{q=1}^{n} q^2 = \frac{n}{6}(n+1)(2n+1) \quad \forall\, n \in \mathbb{N}$:

   Checking $n = 1$:

$$
\sum_{q=1}^{1} q^2 = 1 = \frac{1}{6}(1+1)(2+1). \qquad \checkmark \tag{28}
$$

   Checking $n \Rightarrow n+1$:

$$
\begin{aligned}
\sum_{q=1}^{n+1} q^2 = \sum_{q=1}^{n} q^2 + (n+1)^2 &= \frac{n}{6}(n+1)(2n+1) + (n+1)^2 \\
&= \frac{n+1}{6}[n(2n+1) + 6(n+1)] = \frac{n+1}{6}[2n^2 + 7n + 6] \\
&= \frac{n+1}{6}(2n+3)(n+2) = \frac{n+1}{6}[2(n+1)+1]\,[(n+1)+1]. \qquad \checkmark
\end{aligned}
\tag{29}
$$

2. $\sum_{q=1}^{n} q = \frac{n}{2}(n+1) \quad \forall\, n \in \mathbb{N}$:

   Checking $n = 1$:

$$
\sum_{q=1}^{1} q = 1 = \frac{1}{2}(1+1). \qquad \checkmark \tag{30}
$$

   Checking $n \Rightarrow n+1$:

$$
\begin{aligned}
\sum_{q=1}^{n+1} q = \sum_{q=1}^{n} q + n + 1 &= \frac{n}{2}(n+1) + n + 1 \\
&= \frac{n+1}{2}(n+2) = \frac{n+1}{2}[(n+1)+1]. \qquad \checkmark
\end{aligned}
\tag{31}
$$

Applying these identities to the sum in the last term in eq. (27), we can simplify,

$$\begin{aligned}\sum_{p=1}^{n}(p^2-np) &= \frac{n}{6}(n+1)(2n+1) - n\,\frac{n}{2}(n+1) = \frac{n}{6}(n+1)\Big[2n+1-3n\Big]\\ &= -\frac{n}{6}(n+1)(n-1) = -\frac{n}{6}(n^2-1).\end{aligned} \tag{32}$$

Inserting this result into eq. (27), we arrive at the commutator of two quantum Virasoro generators,

$$\begin{aligned}[L_m, L_n] &= (m-n)L_{m+n} - \frac{D}{2}\frac{n}{6}(n^2-1)\,\delta_{m,-n}\\ &= (m-n)L_{m+n} + \frac{D}{12}\,m(m^2-1)\,\delta_{m,-n}.\end{aligned} \tag{33}$$

2 The second excited level ghost

Compute

$$\langle\phi_2|\phi_2\rangle = \frac{2c_1^2}{25}(D-1)(26-D) \tag{34}$$

for

$$|\phi_2\rangle = \Big(c_1\,\boldsymbol{\alpha}_{-1}^2 + c_2\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-2} + c_3\,(\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})^2\Big)|0,p\rangle. \tag{35}$$

Hint: Given $(L_0-1)|\phi_2\rangle = L_1|\phi_2\rangle = L_2|\phi_2\rangle = 0$ (setting $a=1$), determine the relation between c_1, c_2 and c_3 defining $|\phi_2\rangle$. Then compute $\langle\phi_2|\phi_2\rangle$.

Our goal in this exercise is to evaluate the Virasoro constraints[1]

$$(L_m - a\,\delta_{m,0})|\phi\rangle = 0 \qquad \forall\, m\geq 0 \text{ and } \forall\,|\phi\rangle\in\mathcal{H}_{\text{phys}}, \tag{36}$$

to deduce that $D\leq 26$ is a necessary (but not sufficient) condition for a ghost-free theory. To arrive at this conclusion, it is enough to consider the specific state given in eq. (35).[2] By assuming $|\phi_2\rangle$ to constitute a physical state and setting the normal ordering constant $a=1$, we get exactly the constraints mentioned in the hint. To unravel, what implications these constraints hold for the allowed values of D and the c_i, $i\in\{1,2,3\}$, our strategy will be to commute the Virasoro generators through all of the creation operators in $|\phi_2\rangle$ and let them act directly on $|0,p\rangle$.
First, using $XY = YX + [X,Y]$ on $L_m|\phi_2\rangle$, we get

$$\begin{aligned}L_m|\phi_2\rangle &= \Big(c_1\,\boldsymbol{\alpha}_{-1}^2 L_m + c_1[L_m,\boldsymbol{\alpha}_{-1}^2] + c_2\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-2}L_m + c_2[L_m,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-2}]\\ &\qquad + c_3\,(\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})^2 L_m + c_3[L_m,(\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})^2]\Big)|0,p\rangle\\ &= \frac{1}{2}\delta_{m,0}\,\boldsymbol{\alpha}_0^2|\phi_2\rangle + \Big(c_1[L_m,\boldsymbol{\alpha}_{-1}^2] + c_2\,\boldsymbol{p}\cdot[L_m,\boldsymbol{\alpha}_{-2}] + c_3[L_m,(\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})^2]\Big)|0,p\rangle.\end{aligned} \tag{38}$$

[1] These constraints originated all the way back from $T_{ab}=0$, which arose as the e.o.m. of the worldsheet metric $h_{ab}=0$. Since the Virasoro generators are nothing but the Fourier modes of the energy-momentum tensor, the constraint of having to vanish passes directly on to them.

[2] As opposed to the most general state $|\phi_2\rangle$ at second excited level which can be written i.t.o. the string field modes as

$$|\phi_2\rangle = \Big(\zeta_{\mu\nu}\,\alpha_{-1}^{\mu}\alpha_{-1}^{\nu} + \eta_\mu\alpha_{-2}^{\mu}\Big)|0,p\rangle. \tag{37}$$

where in the last step, we used that L_m for $m \geq 0$ acts on the vacuum as

$$
\begin{aligned}
L_m|0,p\rangle &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \mathcal{N}(\boldsymbol{\alpha}_{m-n}\cdot\boldsymbol{\alpha}_n)|0,p\rangle \\
&= \frac{1}{2}\sum_{n>\lceil\frac{m}{2}\rceil} \boldsymbol{\alpha}_{m-n}\cdot\underbrace{\boldsymbol{\alpha}_n|0,p\rangle}_{0} + \frac{1}{2}\sum_{n=\lfloor\frac{m}{2}\rfloor}^{\lceil\frac{m}{2}\rceil} \mathcal{N}(\boldsymbol{\alpha}_{m-n}\cdot\boldsymbol{\alpha}_n)|0,p\rangle + \frac{1}{2}\sum_{n<\lfloor\frac{m}{2}\rfloor} \boldsymbol{\alpha}_n\cdot\underbrace{\boldsymbol{\alpha}_{m-n}|0,p\rangle}_{0} \\
&= \begin{cases} \frac{1}{2}\boldsymbol{\alpha}_0\cdot\boldsymbol{\alpha}_0|0,p\rangle & \text{if } m=0, \\ \frac{1}{2}\boldsymbol{\alpha}_{m-\frac{m}{2}}\cdot\boldsymbol{\alpha}_{\frac{m}{2}}|0,p\rangle = 0 & \text{if } m\in 2\mathbb{N}, \\ \frac{1}{2}\Big(\boldsymbol{\alpha}_{m-\lceil\frac{m}{2}\rceil}\cdot\boldsymbol{\alpha}_{\lceil\frac{m}{2}\rceil} + \boldsymbol{\alpha}_{\lfloor\frac{m}{2}\rfloor}\cdot\boldsymbol{\alpha}_{m-\lfloor\frac{m}{2}\rfloor}\Big)|0,p\rangle = 0 & \text{if } m\in 2\mathbb{N}+1, \end{cases} \\
&= \frac{1}{2}\delta_{m,0}\,\boldsymbol{\alpha}_0^2|0,p\rangle
\end{aligned}
\tag{39}
$$

Recall that modes α_m^μ with $m \leq 0$ were chosen as the creation operators whereas $m > 0$ corresponded to annihilators.
We will now calculate each commutator the last line of eq. (38) in turn by reusing the identities

$$
[\alpha_m^\mu, \alpha_n^\nu] = m\,\eta^{\mu\nu}\delta_{m,-n} \qquad \text{and} \qquad [\alpha_m^\mu, L_n] = m\alpha_{m+n}^\mu,
\tag{40}
$$

which have already proven useful in exercise 1.
Using $[X, YZ] = [X,Y]Z + Y[X,Z]$, the first commutator becomes

$$
\begin{aligned}
[L_m, \boldsymbol{\alpha}_{-1}^2] &= \underbrace{[L_m,\boldsymbol{\alpha}_{-1}]}_{-(-1)\boldsymbol{\alpha}_{-1+m}}\cdot\boldsymbol{\alpha}_{-1} + \boldsymbol{\alpha}_{-1}\cdot\underbrace{[L_m,\boldsymbol{\alpha}_{-1}]}_{-(-1)\boldsymbol{\alpha}_{-1+m}} = \boldsymbol{\alpha}_{m-1}\cdot\boldsymbol{\alpha}_{-1} + \boldsymbol{\alpha}_{-1}\cdot\boldsymbol{\alpha}_{m-1} \\
&= 2\,\boldsymbol{\alpha}_{m-1}\cdot\boldsymbol{\alpha}_{-1} + \underbrace{[\alpha_{-1,\mu}, \alpha_{m-1}^\mu]}_{-\eta^\mu{}_\mu\delta_{-1,1-m}} = 2\,\boldsymbol{\alpha}_{m-1}\cdot\boldsymbol{\alpha}_{-1} - D\delta_{m,2}.
\end{aligned}
\tag{41}
$$

The second commutator is simpler,

$$
[L_m, \boldsymbol{\alpha}_{-2}] = -(-2)\boldsymbol{\alpha}_{-2+m} = 2\,\boldsymbol{\alpha}_{m-2},
\tag{42}
$$

whereas the third commutator is again a little work,

$$
\begin{aligned}
[L_m, (\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})^2] &= [L_m, \boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}]\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1} + \boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}\,[L_m, \boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}] \\
&= \boldsymbol{p}\cdot[L_m,\boldsymbol{\alpha}_{-1}]\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1} + \boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}\,\boldsymbol{p}\cdot[L_m,\boldsymbol{\alpha}_{-1}] \\
&\overset{(40)}{=} \boldsymbol{p}\cdot\boldsymbol{\alpha}_{m-1}\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1} + \boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{m-1} \\
&= 2\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{m-1}\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1} + \underbrace{[\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}, \boldsymbol{p}\cdot\boldsymbol{\alpha}_{m-1}]}_{-p_\mu p_\nu\eta^{\mu\nu}\delta_{-1,1-m}} \\
&= 2\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{m-1}\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1} - \boldsymbol{p}^2\delta_{m,2}.
\end{aligned}
\tag{43}
$$

Reinserting eqs. (41) to (43) into eq. (38), we get (still with $m \geq 0$),

$$
\begin{aligned}
L_m|\phi_2\rangle = \frac{1}{2}\delta_{m,0}\,\boldsymbol{\alpha}_0^2|\phi_2\rangle + \Big(&2c_1\,\boldsymbol{\alpha}_{m-1}\cdot\boldsymbol{\alpha}_{-1} - c_1 D\,\delta_{m,2} + 2c_2\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{m-2} \\
&+ 2c_3\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{m-1}\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1} - c_3\boldsymbol{p}^2\,\delta_{m,2}\Big)|0,p\rangle.
\end{aligned}
\tag{44}
$$

In particular, considering eq. (44) for $m = 0$ yields

$$
\begin{aligned}
L_0|\phi_2\rangle &= \frac{1}{2}\boldsymbol{\alpha}_0^2|\phi_2\rangle + \Big(2c_1\,\boldsymbol{\alpha}_{-1}\cdot\boldsymbol{\alpha}_{-1} + 2c_2\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-2} + 2c_3\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}\Big)|0,p\rangle \\
&= \Big(\frac{1}{2}\boldsymbol{\alpha}_0^2 + 2\Big)|\phi_2\rangle.
\end{aligned}
\tag{45}
$$

Ergo, the physical state condition $(L_0 - a)|\phi\rangle \overset{!}{=} 0$, where $a = 1$, implies

$$(L_0 - 1)|\phi_2\rangle = \Big(\frac{1}{2}\boldsymbol{\alpha}_0^2 + 1\Big)|\phi_2\rangle \overset{!}{=} 0 \quad \Rightarrow \quad \boldsymbol{\alpha}_0^2 = -2. \tag{46}$$

For $m = 1$ eq. (44) reads

$$L_1|\phi_2\rangle = \Big(2c_1\,\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_{-1} + 2c_2\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_{-1} + 2c_3\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_0\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_{-1}\Big)|0,p\rangle. \tag{47}$$

To proceed here, we need to recall an equality derived in exercise 1.e) on assignment 3, namely $\boldsymbol{\alpha}_0 = \sqrt{2\alpha'}\boldsymbol{p}$. We therefore get

$$\begin{aligned} L_1|\phi_2\rangle &= \Big(2c_1\,\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_{-1} + \frac{2c_2}{\sqrt{2\alpha'}}\,\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_{-1} + \frac{2c_3}{2\alpha'}\,\underbrace{\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_0}_{-2}\,\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_{-1}\Big)|0,p\rangle \\ &= 2\Big(c_1 + \frac{c_2}{\sqrt{2\alpha'}} - \frac{c_3}{\alpha'}\Big)\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_{-1}|0,p\rangle \overset{!}{=} 0 \quad \Rightarrow \quad c_1 + \frac{c_2}{\sqrt{2\alpha'}} - \frac{c_3}{\alpha'} = 0. \end{aligned} \tag{48}$$

Lastly, for $m = 2$, eq. (44) becomes

$$L_2|\phi_2\rangle = \Big(2c_1\,\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_{-1} - c_1 D + 2c_2\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_0 + 2c_3\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_1\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_{-1} - c_3\,\boldsymbol{p}^2\Big)|0,p\rangle. \tag{49}$$

Here, we can apply the mode's commutation relation to let $\boldsymbol{\alpha}_1$ act directly on the vacuum $|0,p\rangle$.

$$\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_{-1}|0,p\rangle = \boldsymbol{\alpha}_{-1} \cdot \underbrace{\boldsymbol{\alpha}_1|0,p\rangle}_{0} + \underbrace{[\alpha_{1,\mu}, \alpha_{-1}^{\mu}]}_{\eta^{\mu}{}_{\mu}\delta_{1,-(-1)}}|0,p\rangle = D|0,p\rangle, \tag{50}$$

$$\boldsymbol{p} \cdot \boldsymbol{\alpha}_1\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_{-1}|0,p\rangle = p_\mu p_\nu \alpha_{-1}^{\mu}\underbrace{\alpha_1^{\nu}|0,p\rangle}_{0} + p_\mu p_\nu \underbrace{[\alpha_1^{\nu}, \alpha_{-1}^{\mu}]}_{\eta^{\mu\nu}\delta_{1,-(-1)}}|0,p\rangle = \boldsymbol{p}^2|0,p\rangle. \tag{51}$$

Reinserting eqs. (50) and (51) into eq. (49) and using $\boldsymbol{p}^2 = \frac{\boldsymbol{\alpha}_0^2}{2\alpha'} = -\frac{1}{\alpha'}$ gives

$$\begin{aligned} L_2|\phi_2\rangle &= \Big(2c_1 D - c_1 D + 2\sqrt{2\alpha'}\,c_2\,\boldsymbol{p}^2 + 2c_3\boldsymbol{p}^2 - c_3\,\boldsymbol{p}^2\Big)|0,p\rangle \overset{!}{=} 0 \\ &\Rightarrow \quad Dc_1 - 2\sqrt{2/\alpha'}\,c_2 - \frac{c_3}{\alpha'} = 0. \end{aligned} \tag{52}$$

We now have two equations for the three state coefficients c_i, $i \in \{1,2,3\}$. We can use them to express c_2 and c_3 i.t.o. c_1. By Inserting eq. (48) into eq. (52), we find

$$Dc_1 - \frac{4c_2}{\sqrt{2\alpha'}} - c_1 - \frac{c_2}{\sqrt{2\alpha'}} = c_1(D-1) - \frac{5c_2}{\sqrt{2\alpha'}} = 0 \quad \Rightarrow \quad c_2 = \frac{\sqrt{2\alpha'}}{5}\,(D-1)\,c_1. \tag{53}$$

Plugging this back into eq. (48) gives

$$c_1 + \frac{1}{5}\,(D-1)\,c_1 - \frac{c_3}{\alpha'} = \frac{1}{5}\,(D+4)\,c_1 - \frac{c_3}{\alpha'} = 0 \quad \Rightarrow \quad c_3 = \frac{D+4}{5}\,\alpha'\,c_1. \tag{54}$$

We are finally in a position to calculate $\langle\phi_2|\phi_2\rangle$ and see what constraints on D we must impose in order to avoid a negative norm of $|\phi_2\rangle$. Note that when expanding the ensuing product of modes, we do not need to consider mixed terms in which $\boldsymbol{\alpha}_{\pm 2}$ appears since it freely commutes with $\boldsymbol{\alpha}_{\pm 1}$ and can thus always act directly on the vacuum. This already reduces the total number of resulting terms from nine to five.

$$\begin{aligned} \langle\phi_2|\phi_2\rangle &= \langle 0,p|\Big(c_1\,\boldsymbol{\alpha}_1^2 + c_2\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_2 + c_3\,(\boldsymbol{p} \cdot \boldsymbol{\alpha}_1)^2\Big)\Big(c_1\,\boldsymbol{\alpha}_{-1}^2 + c_2\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_{-2} + c_3\,(\boldsymbol{p} \cdot \boldsymbol{\alpha}_{-1})^2\Big)|0,p\rangle \\ &= c_1^2\langle 0,p|\boldsymbol{\alpha}_1^2\,\boldsymbol{\alpha}_{-1}^2|0,p\rangle + c_1 c_3\langle 0,p|\boldsymbol{\alpha}_1^2\,(\boldsymbol{p} \cdot \boldsymbol{\alpha}_{-1})^2|0,p\rangle + c_2^2\langle 0,p|\boldsymbol{p} \cdot \boldsymbol{\alpha}_2\,\boldsymbol{p} \cdot \boldsymbol{\alpha}_{-2}|0,p\rangle \\ &\quad + c_3 c_1\langle 0,p|(\boldsymbol{p} \cdot \boldsymbol{\alpha}_1)^2\,\boldsymbol{\alpha}_{-1}^2|0,p\rangle + c_3^2\langle 0,p|(\boldsymbol{p} \cdot \boldsymbol{\alpha}_1)^2\,(\boldsymbol{p} \cdot \boldsymbol{\alpha}_{-1})^2|0,p\rangle. \end{aligned} \tag{55}$$

We'll calculate each of the five contributions in turn.

$$\begin{aligned}\langle 0,p|\boldsymbol{\alpha}_1^2\,\boldsymbol{\alpha}_{-1}^2|0,p\rangle &= \langle 0,p|\alpha_{1,\mu}(\alpha_{-1}^{\nu}\alpha_1^{\mu}+\eta^{\mu\nu})\alpha_{-1,\nu}|0,p\rangle \\ &= \langle 0,p|\alpha_{1,\mu}\alpha_{-1}^{\nu}(\alpha_{-1,\nu}\alpha_1^{\mu}+\eta^{\mu}{}_{\nu})|0,p\rangle + \langle 0,p|\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_{-1}|0,p\rangle \\ &= \langle 0,p|\alpha_{1,\mu}\alpha_{-1}^{\nu}\alpha_{-1,\nu}\underbrace{\alpha_1^{\mu}|0,p\rangle}_{0} + 2\langle 0,p|\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_{-1}|0,p\rangle \overset{(50)}{=} 2D,\end{aligned} \tag{56}$$

$$\begin{aligned}\langle 0,p|\boldsymbol{\alpha}_1^2\,(\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})^2|0,p\rangle &= p_\nu p_\rho\langle 0,p|\alpha_{1,\mu}(\alpha_{-1}^{\nu}\alpha_1^{\mu}+\eta^{\mu\nu})\alpha_{-1}^{\rho}|0,p\rangle \\ &= p_\nu p_\rho\langle 0,p|\alpha_{1,\mu}\alpha_{-1}^{\nu}(\alpha_{-1}^{\rho}\alpha_1^{\mu}+\eta^{\mu\rho})|0,p\rangle + \langle 0,p|\boldsymbol{p}\cdot\boldsymbol{\alpha}_1\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}|0,p\rangle \\ &= p_\nu p_\rho\langle 0,p|\alpha_{1,\mu}\alpha_{-1}^{\nu}\alpha_{-1}^{\rho}\underbrace{\alpha_1^{\mu}|0,p\rangle}_{0} + 2\langle 0,p|\boldsymbol{p}\cdot\boldsymbol{\alpha}_1\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}|0,p\rangle \overset{(51)}{=} 2\boldsymbol{p}^2,\end{aligned} \tag{57}$$

$$\langle 0,p|\boldsymbol{p}\cdot\boldsymbol{\alpha}_2\,\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-2}|0,p\rangle = p_\mu p_\nu\langle 0,p|(\alpha_{-2}^{\nu}\alpha_2^{\mu}+2\eta^{\mu\nu})|0,p\rangle = 2\boldsymbol{p}^2, \tag{58}$$

$$\langle 0,p|(\boldsymbol{p}\cdot\boldsymbol{\alpha}_1)^2\,\boldsymbol{\alpha}_{-1}^2|0,p\rangle = \big(\langle 0,p|\boldsymbol{\alpha}_1^2\,(\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})^2|0,p\rangle\big)^{\dagger} \overset{(57)}{=} (2\boldsymbol{p}^2)^{\dagger} = 2\boldsymbol{p}^2, \tag{59}$$

$$\langle 0,p|(\boldsymbol{p}\cdot\boldsymbol{\alpha}_1)^2\,(\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})^2|0,p\rangle = 2\boldsymbol{p}^4. \tag{60}$$

With eqs. (56) to (60) inserted, eq. (55) reads

$$\begin{aligned}\langle\phi_2|\phi_2\rangle &= 2Dc_1^2 + 2\boldsymbol{p}^2c_1c_3 + 2\boldsymbol{p}^2c_2^2 + 2\boldsymbol{p}^2c_3c_1 + 2\boldsymbol{p}^4c_3^2 \\ &\overset{(53)}{=} 2c_1^2\left[D + 2\boldsymbol{p}^2\frac{D+4}{5}\,\alpha' + \boldsymbol{p}^2\Big(\frac{\sqrt{2\alpha'}}{5}\,(D-1)\Big)^2 + \boldsymbol{p}^4\Big(\frac{D+4}{5}\,\alpha'\Big)^2\right] \\ &= 2c_1^2\left[D - \frac{2D+8}{5} - \frac{2}{25}\,(D-1)^2 + \frac{(D+4)^2}{25}\right] \\ &= \frac{2c_1^2}{25}\Big[25D - 10D - 40 - 2D^2 + 4D - 2 + D^2 + 8D + 16\Big] \\ &= \frac{2c_1^2}{25}\Big[27D - 26 - D^2\Big] = \frac{2c_1^2}{25}(D-1)(26-D) \quad < 0 \quad \text{if } D > 26.\end{aligned} \tag{61}$$

Thus we find that a theory operating in more than 26 dimensions would suffers from negative norm states, so-called ghosts.

3 Some classical lightcone gauge identities

Show that in target-space lightcone gauge the constraint $T_{ab} = 0$ implies

$$\partial_\pm X^- = \frac{l}{2\pi\alpha' p^+}(\partial_\pm X_\perp)^2. \tag{62}$$

Show that for the open string with NN boundary conditions the X^- oscillators can be solved for in terms of the transverse oscillators as

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}p^+}\frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{D-2}\alpha_{n-m}^i\,\alpha_m^i. \tag{63}$$

Note that the sum over oscillators includes $\alpha_0^i = \sqrt{2\alpha'}p^i$.

In exercise 1.b) on assignment 2, we varied the Polyakov action w.r.t. the worldsheet metric h^{ab} to obtain the classical equation of motion $T_{ab} = 0$. Moving to lightcone gauge, we found that the constraints $T_{+-} = T_{-+} = 0$ were automatically satisfied due to the requirement of tracelessness imposed on the energy momentum tensor T_{ab} as a direct consequence of Weyl invariance.[3] The

[3]The exact argument here was that $0 \overset{!}{=} T^a{}_a = h_{ab}T^{ab} = h_{+-}T^{+-} + h_{-+}T^{-+} = 2h_{+-}T^{+-} = 2(-\frac{1}{2})T^{+-} = -T^{+-}$, where we used that $h_{++} = h_{--} = 0$, and that h_{ab} and T_{ab} are both symmetric tensors.

components T_{+-}, T_{-+} were a priori non-vanishing and given by

$$T_{ab} = -\frac{1}{\alpha'}\left[\partial_a X \cdot \partial_b X - \frac{1}{2} h_{ab} h^{cd} \partial_c X \cdot \partial_d X\right] \quad \overset{h_{\pm\pm}=0}{\Longrightarrow} \quad T_{\pm\pm} = -\frac{1}{\alpha'}\partial_\pm X \cdot \partial_\pm X. \tag{64}$$

However, we cannot gauge away dynamical information such as $T_{ab} = 0$ (only redundancies in our physical description are affected by gauge transformations). So if it was true in the original worldsheet coordinates (τ, σ), it must remain true in lightcone coordinates (ξ^+, ξ^-). Thus when working in lightcone gauge, we have to implement as constraints

$$T_{++} = T_{--} = 0. \tag{65}$$

Inserting $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ into eq. (64) and enforcing eq. (65) gives

$$(\partial_\tau X \pm \partial_\sigma X)^2 = 0. \tag{66}$$

This expression becomes useful when we recall that the (flat) ambient space metric $\eta_{\mu\nu}$ in lightcone gauge is given by

$$\left.\begin{aligned} &\eta_{+-} \equiv \eta_{0,D-1} = -1 = \eta_{D-1,0} \equiv \eta_{-+}, \\ &\eta_{ij} = \delta_{ij}, \quad i,j \in \{1, \dots, D-2\}, \end{aligned}\right\} \text{ i.e. } \quad \boldsymbol{\eta} = \begin{pmatrix} 0 & & & & -1 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -1 & & & & 0 \end{pmatrix}, \tag{67}$$

so that the Minkowski product of the string field

$$X^2 = \eta_{\mu\nu} X^\mu X^\nu = -2X^+X^- + \sum_{i=1}^{D-2} (X^i)^2 \equiv -2X^+X^- + X_\perp^2, \tag{68}$$

where $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$. Applying this scheme to eq. (66) yields

$$-2(\partial_\tau X \pm \partial_\sigma X)^+ (\partial_\tau X \pm \partial_\sigma X)^- + (\partial_\tau X \pm \partial_\sigma X)_\perp^2 = 0. \tag{69}$$

The next step is where working in lightcone gauge pays off. Lightcone gauge implies that we are already dealing with a flat worldsheet, i.e. $h_{ab} = \eta_{ab}$. However, there is still left a residual reparametrization invariance[4] generated by the conformal Killing vector fields ϵ_a which fulfill $\nabla_a \epsilon_b + \nabla_b \epsilon_a = h_{ab} \nabla_c \epsilon^c$. We can fix this remaining invariance by transforming into a set of coordinates in which we identify the ambient space dimension X^+ with the worldsheet's time dimension τ:[5]

$$X^+ = \frac{2\pi\alpha'}{l} p^+ \tau + x^+. \tag{70}$$

This transformation leaves us with

$$\partial_\tau X^+ = \frac{2\pi\alpha'}{l} p^+ \qquad \text{and} \qquad \partial_\sigma X^- = 0, \tag{71}$$

which we can insert into eq. (69) to get

$$-2\frac{2\pi\alpha'}{l} p^+ \underbrace{(\partial_\tau X \pm \partial_\sigma X)^-}_{2\partial_\pm X^-} + \underbrace{(\partial_\tau X \pm \partial_\sigma X)_\perp^2}_{4(\partial_\pm X_\perp)^2} = 0 \qquad \Rightarrow \qquad \partial_\pm X^- = \frac{l}{2\pi\alpha' p^+}(\partial_\pm X_\perp)^2. \tag{72}$$

This is the first identity we were asked to derive.

[4] In the lecture notes, this is often referred to as residual conformal symmetry.

[5] This has several advantages unrelated to our line of thought: By fixing the residual reparametrization invariance, all ghosts and unphysical degrees of freedom are eliminated. The disadvantage is that Lorentz covariance becomes non-manifest, i.e. hard to prove.

For the second one, we recall the open string mode expansion for Neumann boundary conditions at both ends as derived in great detail in exercise 1.b) on assignment 3,

$$X^\mu(\tau,\sigma) = x_0^\mu + \frac{2\pi\alpha'}{l} p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-\frac{\pi i}{l} n\tau} \cos\left(\tfrac{\pi}{l} n\sigma\right). \tag{73}$$

Differentiation w.r.t. $\xi^\pm$ yields

$$\begin{aligned}
\partial_\pm X^\mu(\tau,\sigma) &= \frac{1}{2}(\partial_\tau X^\mu \pm \partial_\sigma X^\mu) \\
&= \frac{\pi\alpha'}{l} p^\mu + \frac{\pi}{l}\sqrt{\tfrac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^\mu e^{-\frac{\pi i}{l} n\tau} \cos\left(\tfrac{\pi}{l} n\sigma\right) \mp i\frac{\pi}{l}\sqrt{\tfrac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^\mu e^{-\frac{\pi i}{l} n\tau} \sin\left(\tfrac{\pi}{l} n\sigma\right) \\
&= \frac{\pi\alpha'}{l} p^\mu + \frac{\pi}{l}\sqrt{\tfrac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^\mu e^{-\frac{\pi i}{l} n\tau} e^{\mp\frac{\pi i}{l} n\sigma} \\
&= \frac{\pi}{l}\sqrt{\tfrac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-\frac{\pi i}{l} n\xi^\pm},
\end{aligned} \tag{74}$$

where we used $\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$ in the last step to consolidate the $n = 0$-term back into the sum. This is indeed the original Fourier series we started with in exercise 1.b) when deriving the string field's mode expansion and thus a good check for consistency.
Inserting this series into relation (72) between X^μ's lightcone and orthogonal components, we get

$$\begin{aligned}
\partial_\pm X^- &= \frac{\pi}{l}\sqrt{\tfrac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \alpha_m^- e^{-\frac{\pi i}{l} m\xi^\pm} \\
&\overset{!}{=} \frac{l}{2\pi\alpha' p^+} \frac{\pi^2}{l^2} \frac{\alpha'}{2} \sum_{i=1}^{D-2} \sum_{m\in\mathbb{Z}} \alpha_m^i e^{-\frac{\pi i}{l} m\xi^\pm} \sum_{k\in\mathbb{Z}} \alpha_k^i e^{-\frac{\pi i}{l} k\xi^\pm} = \frac{l}{2\pi\alpha' p^+} (\partial_\pm X_\perp)^2.
\end{aligned} \tag{75}$$

By multiplying both sides of the above equation with $e^{-\frac{\pi i}{l} n\xi^\pm}$ and integrating $\xi^\pm$ from $-l$ to l, this simplifies to

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \alpha_m^- \underbrace{\int_{-l}^{l} \mathrm{d}\xi^\pm\, e^{\frac{\pi i}{l}(n-m)\xi^\pm}}_{2l\,\delta_{n,m}} &= \frac{l}{2\pi\alpha' p^+} \frac{\pi}{l}\sqrt{\tfrac{\alpha'}{2}} \sum_{i=1}^{D-2} \sum_{k,m\in\mathbb{Z}} \alpha_k^i \alpha_m^i \underbrace{\int_{-l}^{l} \mathrm{d}\xi^\pm\, e^{\frac{\pi i}{l}(n-m-k)\xi^\pm}}_{2l\,\delta_{n-m,k}} \\
\Rightarrow \alpha_n^- &= \frac{1}{\sqrt{2\alpha'} p^+} \frac{1}{2} \sum_{i=1}^{D-2} \sum_{m\in\mathbb{Z}} \alpha_{n-m}^i \alpha_m^i.
\end{aligned} \tag{76}$$

Thus for the open string with Neumann boundary conditions, all those lightcone modes α_n^- that were not gauged into oblivion by (70) can be expressed in terms of the transverse modes α_n^i, $i \in \{1,\ldots,D-2\}$. From this fact, we can infer that the lightcone dimensions don't actually contain any physical degrees of freedom but are pure gauge material.