

Quantum Field Theory II - Assignment 4Problem 4.1 (The 1PI effective action)

In the lecture, we have defined

$$T(x_1, \dots, x_n) = \frac{\delta}{\delta j(x_1)} \dots \frac{\delta}{\delta j(x_n)} iW[j], \quad (1)$$

$$\tilde{\Gamma}(x_1, \dots, x_n) = \frac{\delta}{\delta \varphi(x_1)} \dots \frac{\delta}{\delta \varphi(x_n)} \Gamma[\varphi]. \quad (2)$$

What is the interpretation of $T(x_1, \dots, x_n)$ and $\Gamma(x_1, \dots, x_n)$?

We have presented both a computational and a graphic method to relate $T(x_1, x_2, x_3)$ and $\tilde{\Gamma}(x_1, x_2, x_3)$. Revise these methods and use them to relate $T(x_1, x_2, x_3, x_4)$ and $\tilde{\Gamma}(x_1, x_2, x_3, x_4)$. Argue that the result demonstrates that $\Gamma[\varphi]$ is indeed a generating functional of the 1PI-correlators.

In analogy to $G_j(x_1, \dots, x_n) = \frac{\delta^n}{\delta j(x_1) \dots \delta j(x_n)} \frac{Z[j]}{Z[0]}$, where $\frac{Z[j]}{Z[0]}$ is the generating functional for all n -point Green's functions $G_j(x_1, \dots, x_n)$ in the presence of the source j , $T(x_1, \dots, x_n) = G_j^{(c)}(x_1, \dots, x_n) = \frac{\delta^n}{\delta j(x_1) \dots \delta j(x_n)} iW[j]$ denotes a fully connected n -point Green's function in the presence of j and $iW[j]$ is called the effective action. The fully connected are a subset of all Green's functions. An even stronger reduction are the fully connected, amputated 1PI Green's functions $\tilde{\Gamma}(x_1, \dots, x_n)$ in the presence of the field φ (or without $\tilde{\Gamma}(x_1, \dots, x_n)|_{\varphi=0}$) generated by the 1PI effective action $\Gamma[\varphi]$.

In the case of $n=3$, $T(x_1, x_2, x_3)$ and $\tilde{\Gamma}(y_1, y_2, y_3)$ are related as follows,

$$T(x_1, x_2, x_3) = \int d^4y_1 d^4y_2 d^4y_3 T(x_1, y_1) T(x_2, y_2) T(x_3, y_3) \tilde{\Gamma}(y_1, y_2, y_3)$$

or displayed graphically

$$T_3(x_1, x_2, x_3) = \textcircled{W} = x_1 \textcircled{W} \textcircled{W} x_2 \textcircled{W} \textcircled{\Gamma} \textcircled{W} x_3$$

$$T(x_1, x_2, x_3, x_4) = \frac{\delta}{16 \int_{\mathcal{X}(4)} T(x_1, x_2, x_3) = \frac{5}{16 \int_{\mathcal{X}(4)} \int_{\mathcal{X}} \int_{\mathcal{X}} \int_{\mathcal{X}} T(x_1, y_1) T(x_2, y_2) T(x_3, y_3) T(x_4, y_4) dy_1 dy_2 dy_3 dy_4}$$
$$\frac{\delta}{i\delta\varphi(x_0)} = \int d^4y_1 \frac{\delta\varphi(y_1)}{i\delta\varphi(y_0)} \frac{\delta}{\delta\varphi(y_1)} = \int d^4y \frac{\delta^2 W[\varphi]}{i\delta\varphi(x_0) i\delta\varphi(y_1)} \frac{\delta}{\delta\varphi(y_1)} = \int d^4x \pi_2(x_0, y_1) \frac{\delta}{\delta\varphi(y_1)}$$
$$\begin{aligned} \Gamma(x_1, x_2, x_3, x_4) &= \int_{k=1}^4 d^4 y \Gamma(x_k, y_k) \frac{\frac{\delta}{\delta y_k} \Gamma(y_1, y_2, y_3)}{\Gamma(y_1, y_2, y_3, x_4)} + \int_{k=1}^4 d^4 y_k \Gamma(x_1, y_1, y_2) \Gamma(x_2, x_3) \Gamma(x_4, y_4) \tilde{\Gamma}(y_1, y_2, y_3) \\ &+ \int_{k=1}^4 d^4 y_k \Gamma(x_1, y_1) \Gamma(x_2, y_2, y_3) \Gamma(x_4, y_4) \tilde{\Gamma}(y_1, y_2, y_3) - \int_{k=1}^4 d^4 y_k \Gamma(x_1, y_1) \Gamma(x_2, y_1) \Gamma(x_3, y_2, y_3) \tilde{\Gamma}(y_1, y_2, y_3) \end{aligned}$$

The diagram illustrates the decomposition of a star graph S_{n-1} into a sum of products of smaller star graphs. On the left, a star graph is shown with a central node W_1 and $n-1$ leaf nodes $x_1, x_2, \dots, x_{n-1}$. This is equal to the sum of products of star graphs, each with a central node W_i and a path of nodes $x_1, x_2, \dots, x_{n-1}$.

gives indeed the amputated, fully connected 1PI 4-point Green's function.

resummed Propagator $G_2 = \text{---} \bigcirc \text{---} \Big|_{j=0} = \text{---} \bigcirc \text{---}$, and the tree

Therefore, the fourth derivative of $\mathcal{J}[\varphi]$ wr.t. φ is exactly the 1PI 4-point function.

Problem 4.2 (Coleman-Weinberg potential for ϕ^4 -theory)

In this exercise, we calculate the quantum effective action $\Gamma[\varphi]$ for the ϕ^4 -theory,

$$S[\phi] = \int d^4x \left(-\frac{1}{2} \phi(x) \partial_\mu^2 \phi(x) + \tilde{m}^2 \phi(x) - \frac{\lambda}{4!} \phi^4(x) \right), \quad \tilde{m}^2 = m^2 - i\epsilon \quad (3)$$

up to one-loop order for a constant background field $\varphi = \varphi_0$. In the lecture, it was shown that

$$\Gamma[\varphi] = S[\varphi] + \hbar X^{1\text{-loop}}[\varphi] + \text{'higher loop corrections'}, \quad \text{with} \quad (4)$$

$$X^{1\text{-loop}}[\varphi] = -i \ln \int \mathcal{D}\phi \exp \left(i \int d^4x \left(-\frac{1}{2} \phi \partial_\mu^2 \phi + \tilde{m}^2 \phi - \frac{\lambda}{4!} \phi^4 \right) \right), \quad (5)$$

where the measure $\mathcal{D}$ is defined such that $\mathcal{Z}[\mathcal{D}] = 1$.

a) Show that for any Hermitian matrix A , $\ln(\det A) = \text{Tr}(\ln A)$.

Hint: You may find it useful to perform a basis change for which A becomes diagonal.

Any complex normal matrix (satisfying $A^\dagger A = A A^\dagger$) has an orthogonal eigenvector basis. Assembling these eigenvectors into the unitary matrix U ($U U^\dagger = U^\dagger U = \mathbb{1}$), any complex normal matrix A can be decomposed

$$A = U D U^\dagger,$$

where D is a diagonal matrix which, if A is also Hermitian ($A = A^\dagger$), has only real values, namely the eigenvalues λ_i of A .

Performing such a change of basis, we get

$$\begin{aligned} \ln(\det A) &= \ln(\det(U D U^\dagger)) = \ln(\det U \cdot \det D \cdot \det U^\dagger) \\ &= \ln(\det(U U^\dagger) \prod_i \lambda_i) = \sum_i \ln(\lambda_i) = \text{Tr}(\ln A) \end{aligned}$$

b) Evaluate K^{1-loop} by means of the path integral techniques presented in assignments 2 and 3. In analogy to part a), use that as for a differential operator D , $\ln(\det D) = \text{Tr}(\ln D)$. How do you interpret the trace?

Hint: The result can be found in the lecture notes.

$$K^{1-loop}[\varphi] = -i n \int \mathcal{D}\phi e^{-\frac{i}{2} f(\varphi) + \frac{i}{2} \frac{\delta^2 S(\varphi)}{\delta \phi \delta \phi}},$$

where $\int \mathcal{D}\phi := \frac{1}{Z[0]} \int \mathcal{D}\phi = \frac{1}{Z_0[\varphi_0 + \sigma(\varphi)]} \int \mathcal{D}\phi$. Since we are performing here a perturbative expansion in powers of $\hbar$, where $K^{1-loop}[\varphi]$ corresponds to order $\hbar^1$ whereas $\sigma(\varphi)$ contains higher powers of $\hbar$, we drop the latter and only carry along for normalisation the generating functional $Z_0[\varphi]$ of the free theory at zero source J , given by

$$Z_0[\varphi] = \int \mathcal{D}\phi e^{i S_0[\varphi]}, \quad \text{with } S_0[\varphi] = S[\varphi]|_{J=0}.$$

Before performing the path integral, we explicitly calculate the exponent

$$\begin{aligned} \frac{i}{2\hbar} \frac{\delta^2 S(\varphi)}{\delta \varphi(x) \delta \varphi(y)} &= \frac{-i}{2\hbar} \frac{\delta}{\delta \varphi(x)} \frac{\delta}{\delta \varphi(y)} \int d^4x' \left(-\frac{1}{2} \phi(x') (\partial_{\mu}^2 + \tilde{m}^2) \phi(x') - \frac{\lambda}{4!} \phi^4(x') \right) \\ &\quad - (\partial_x^2 + \tilde{m}^2) \delta(x-y), \quad \text{see eq. (5), assignment 2} \\ &= \frac{-i}{2\hbar} \left[((\partial_x^2 + \tilde{m}^2) \delta(x-y) + \frac{\lambda}{2} \phi(x) \delta(x-y)) \right] =: D. \end{aligned}$$

Now, inserting D back into our expression for $K^{1-loop}[\varphi]$ and using our result from exercise 3.2 d), more specifically

$$\int \mathcal{D}\phi e^{\phi \cdot D \cdot \phi} = \sqrt{\frac{i^n}{\det D}}, \quad \text{for some differential operator } D,$$

where n is simply the dimensionality of the integral measure, we can carry out the path integration for a static background field $\varphi = \phi_0$ to arrive at

$$K^{1-loop}[\varphi_0] = -i\hbar \frac{\int \mathcal{D}\phi e^{-\phi \cdot D \cdot \phi}}{\int \mathcal{D}\phi e^{-\phi \cdot D_{\phi_0} \cdot \phi}} \Big|_{\phi=\phi_0} = -i\hbar \ln \frac{\sqrt{\frac{i^n}{\det D}}}{\sqrt{\frac{i^n}{\det D_{\phi_0}}}} = -\frac{i}{2} \ln \left(\frac{\det D_{\phi_0}}{\det D} \right) \Big|_{\phi=\phi_0}$$

Using now the expression we showed in part a), we can simplify further

$$\begin{aligned} X^{1-loop}[\phi_0] &= \frac{i}{2} \ln \left(\frac{\det D_{\phi_0}}{\det D} \right) \Big|_{\phi_0=\phi_0} = -\frac{i}{2} \left(\ln(\det D_{\phi_0}) - \ln(\det D_{\phi_0}) \right) \\ &= -\frac{i}{2} \left(\text{Tr}(\ln D_{\phi_0}) - \text{Tr}(\ln D_{\phi_0}) \right) = -\frac{i}{2} \text{Tr} \left(\ln \left(\frac{D_{\phi_0}}{D_{\phi_0}} \right) \right) \\ &= -\frac{i}{2} \text{Tr} \left(\ln \left(\frac{\frac{i}{2k}(\partial_x^2 + m^2) \delta(x-y)}{\frac{i}{2k}(\partial_x^2 + m^2) + \frac{i}{2} \phi_0^2 \delta(x-y)} \right) \right) = -\frac{i}{2} \text{Tr} \left(\ln \left(\frac{\partial_x^2 + m^2}{\partial_x^2 + m^2 + \frac{1}{2} \phi_0^2} \right) \right) \end{aligned}$$

The trace of a differential operator is given by the sum of its eigenvalues, for example a common operator trace appearing in statistical physics is $Z = \text{Tr} e^{-\beta H} = \sum_n \langle n | e^{-\beta H} | n \rangle = \sum_n \langle n | e^{-\beta E_n} | n \rangle = \sum_n e^{-\beta E_n}$.

A suitable eigenbasis for ∂_x^2 is e^{ikx} , however since D 's eigenvalues are continuous rather than discrete, we need to integrate over the basis

$$\begin{aligned} X^{1-loop}[\phi_0] &= \frac{i}{2} \text{Tr} \left(\ln \left(\frac{\partial_x^2 + m^2}{\partial_x^2 + m^2 + \frac{1}{2} \phi_0^2} \right) \right) = \int d^4x \int \frac{d^4k}{(2\pi)^4} \underbrace{\langle k | x \rangle}_{\frac{1}{\sqrt{2\pi}} e^{ikx}} \frac{i}{2} \ln \left(\frac{\partial_x^2 + m^2}{\partial_x^2 + m^2 + \frac{1}{2} \phi_0^2} \right) \underbrace{\langle x | k \rangle}_{\frac{1}{\sqrt{2\pi}} e^{-ikx}} \\ &= \frac{i}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} \ln \left(\frac{-k^2 + m^2 + \frac{1}{2} \phi_0^2}{-k^2 + m^2} \right) e^{i(kx-kx)} = \frac{i}{2} V_0 \int \frac{d^4k}{(2\pi)^4} \ln \left(\frac{m^2 + \frac{1}{2} \phi_0^2 - k^2}{m^2 - k^2} \right) \end{aligned}$$

where we used the Taylor series representation of the logarithm;

$$\ln(2^x) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (-1 + 2^x)^n,$$

to let ∂_x^2 act directly on its eigenstates.

c) We define the effective potential $V(\varphi) = -V_{0, \text{eff}}[\Gamma(\varphi)]$. Give the potential up to one loop-order, $V^{\text{tree}}(\varphi = \varphi_0) + V^{1\text{-loop}}(\varphi = \varphi_0)$.

Hint: Starting from the expression found in part b), you should find an integral representation of $V^{1\text{-loop}}(\varphi = \varphi_0)$ of the form

$$\int dk (\sqrt{k^2 + X + \delta} - \sqrt{k^2 + X}). \quad (6)$$

To this end, perform the integral over k^0 as a suitable contour integral.

Furthermore, the following integral identity might be useful

$$\int_x^{x+\delta} d\xi \frac{1}{\xi} = \ln\left(\frac{x+\delta}{x}\right).$$

Give a physical interpretation of the two terms in eq. (6).

Considering definitions given in the script, i.e. eqs. (7.19f) - (7.20f), we deduce the following:

$$\Gamma[\varphi] := S[\varphi] + \hbar K[\varphi], \quad \text{with } K[\varphi] = K^{1\text{-loop}}[\varphi] + \mathcal{O}(\hbar^2), \quad \text{and}$$

$$\Gamma[\varphi] = -V_{0, \text{eff}} V(\varphi) = -V_{0, \text{eff}} (V^0(\varphi) + V^{1\text{-loop}}(\varphi) + \dots)$$

$$\Rightarrow V^{1\text{-loop}}(\varphi_0) = -\frac{\hbar}{V_{0, \text{eff}}} K^{1\text{-loop}}[\varphi_0] = \frac{\hbar}{2i} \int \frac{d^4 k}{(2\pi)^4} \ln\left(\frac{m^2 + \frac{1}{2}\phi_0^2 - k^2 - i\epsilon}{m^2 - k^2 - \epsilon}\right)$$

Using the hinted at integral identity backwards, we can rewrite the above,

$$V^{1\text{-loop}}(\varphi_0) = \frac{-\hbar}{2i} \int \frac{d^4 k}{(2\pi)^4} \int_{m^2}^{m^2 + \frac{1}{2}\phi_0^2} d\mu^2 \frac{1}{k^2 - \mu^2 - i\epsilon} = \frac{-\hbar}{2i} \int \frac{d^4 k}{(2\pi)^4} \int_{\mathcal{C}} \frac{d\mu^2}{2\pi} \frac{1}{k^2 - \mu^2 - i\epsilon} \frac{1}{\mu^2 - \sqrt{k^2 + \mu^2 + i\epsilon}},$$

where $\mathcal{C}$ is a path along the real axis closed at infinity by a half circle in the upper complex plane, thereby picking up the pole at $k^0 = \sqrt{k^2 + \mu^2 + i\epsilon}$ when applying the residue theorem to perform the k^0 -integral. We pick up the contribution,

$$\text{Res}\left(\frac{1}{\mu^2 - \sqrt{k^2 + \mu^2 + i\epsilon}}, \sqrt{k^2 + \mu^2 + i\epsilon}\right) = \lim_{k^0 \rightarrow \sqrt{k^2 + \mu^2 + i\epsilon}} (k^0 - \sqrt{k^2 + \mu^2 + i\epsilon}) \frac{1}{k^0 - \sqrt{k^2 + \mu^2 + i\epsilon}} = 1,$$

and no change in sign, because C is positively oriented, so that

$$\begin{aligned}
 V^{-1}(\varphi) &= \frac{-\hbar}{2i} \int_{\frac{m^2}{2} - \frac{\Delta^2}{2}}^{\frac{m^2}{2} + \frac{\Delta^2}{2}} \frac{d\mu^2}{\exp} \frac{1}{2\pi} 2\pi i \frac{1}{\sqrt{k^2 + \mu^2 + i\epsilon} + \sqrt{k^2 + \mu^2 - i\epsilon}} \stackrel{C \rightarrow 0}{=} -\frac{\hbar}{4} \int_{\frac{m^2}{2} - \frac{\Delta^2}{2}}^{\frac{m^2}{2} + \frac{\Delta^2}{2}} \frac{d\mu^2}{\sqrt{k^2 + \mu^2}} \\
 &= -\frac{\hbar}{4} \int_{\frac{m^2}{2} - \frac{\Delta^2}{2}}^{\frac{m^2}{2} + \frac{\Delta^2}{2}} \frac{d\mu^2}{(\mu^2)} 2\sqrt{k^2 + \mu^2} \Big|_{\frac{m^2}{2} - \frac{\Delta^2}{2}}^{\frac{m^2}{2} + \frac{\Delta^2}{2}} = -\frac{\hbar}{2} \int_{\frac{m^2}{2} - \frac{\Delta^2}{2}}^{\frac{m^2}{2} + \frac{\Delta^2}{2}} \frac{d\mu^2}{\mu^2} \left(\sqrt{k^2 + \mu^2 + \frac{\Delta^2}{2}} - \sqrt{k^2 + \mu^2} \right)
 \end{aligned}$$