

String Theory

Solution to Assignment 4

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1 The quantum Virasoro algebra

In this exercise, we will show that in the quantized bosonic string theory the normal ordered Virasoro generators

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathcal{N}(\alpha_{m-n} \cdot \alpha_n) \quad (1)$$

satisfy the Virasoro algebra with central charge^a

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m,-n}. \quad (2)$$

In order to become more familiar with the normal ordering prescription, we will do this by brute force methods, i.e., by simply using the definition of the normal ordered generators L_m and then calculating their commutators. We will proceed in several smaller steps.

- a) Explain why the normal ordering in L_m only affects L_0 and why the Virasoro generators L_m can be written in the following form,

$$L_m = \frac{1}{2} \sum_{n=-\infty}^0 \alpha_n \cdot \alpha_{m-n} + \frac{1}{2} \sum_{n=1}^{\infty} \alpha_{m-n} \cdot \alpha_n. \quad (3)$$

- b) Using $[X, YZ] = [X, Y]Z + Y[X, Z]$ and $[\alpha_m^\mu, \alpha_n^\nu] = m \eta^{\mu\nu} \delta_{m,-n}$, prove that for all $m, n \in \mathbb{Z}$,

$$[\alpha_m^\mu, L_n] = m \alpha_{m+n}^\mu. \quad (4)$$

- c) Decompose the sum

$$\sum_{n=-\infty}^{\infty} = \sum_{n=-\infty}^0 + \sum_{n=1}^{\infty} \quad (5)$$

as we did in eq. (3) to “solve” the normal ordering condition. Use the result of part b) to show that

$$\begin{aligned} [L_m, L_n] &= \frac{1}{2} \sum_{l=-\infty}^0 [(m-l) \alpha_l \cdot \alpha_{m+n-l} + l \alpha_{n+l} \cdot \alpha_{m-l}] \\ &\quad + \frac{1}{2} \sum_{l=1}^{\infty} [(m-l) \alpha_{m+n-l} \cdot \alpha_l + l \alpha_{m-l} \cdot \alpha_{n+l}]. \end{aligned} \quad (6)$$

d) Make the substitution $p = n + l$ in the second and fourth term in eq. (6) and verify

$$[L_m, L_n] = \frac{1}{2} \sum_{l=-\infty}^0 (m-l) \alpha_l \cdot \alpha_{m+n-l} + \frac{1}{2} \sum_{p=-\infty}^n (p-n) \alpha_p \cdot \alpha_{m+n-p} \\ + \frac{1}{2} \sum_{l=1}^{\infty} (m-l) \alpha_{m+n-l} \cdot \alpha_l + \frac{1}{2} \sum_{p=n+1}^{\infty} (p-n) \alpha_{m+n-p} \cdot \alpha_p. \quad (7)$$

e) From now on, we will restrict ourselves to the case $n > 0$, as the other cases $n < 0$ and $n = 0$ are completely analogous. Show, therefore, that, for $n > 0$, the expression eq. (7) in part d) is equal to

$$[L_m, L_n] = \frac{1}{2} \sum_{p=-\infty}^0 (m-n) \alpha_p \cdot \alpha_{m+n-p} + \frac{1}{2} \sum_{p=1}^n (p-n) \alpha_p \cdot \alpha_{m+n-p} \\ + \frac{1}{2} \sum_{p=n+1}^{\infty} (m-n) \alpha_{m+n-p} \cdot \alpha_p + \frac{1}{2} \sum_{p=1}^n (m-p) \alpha_{m+n-p} \cdot \alpha_p. \quad (8)$$

Which of these terms are already normal-ordered?

f) Prove

$$\sum_{p=1}^n (p-n) \alpha_p \cdot \alpha_{m+n-p} = \sum_{p=1}^n (p-n) \alpha_{m+n-p} \cdot \alpha_p + \sum_{p=1}^n (p-n) p D \delta_{m,-n}, \quad (9)$$

and insert this for the second term in eq. (8) of part e).

g) Show that your result from part e) is now equivalent to

$$[L_m, L_n] = \frac{1}{2} \sum_{l=-\infty}^{\infty} (m-n) \mathcal{N}(\alpha_l \cdot \alpha_{m+n-l}) + \frac{1}{2} D \sum_{l=1}^n (l^2 - nl) \delta_{m,-n}. \quad (10)$$

h) Prove, e.g. by induction, the following identities,

$$\sum_{q=1}^n q^2 = \frac{n}{6} (n+1)(2n+1), \quad (11)$$

$$\sum_{q=1}^n q = \frac{n}{2} (n+1), \quad (12)$$

and use this to finally derive

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12} m(m^2 - 1) \delta_{m,-n}. \quad (13)$$

from the expression in part g).

^aA central charge, T_0 , of a Lie algebra is a generator that commutes with all generators of the Lie algebra, $[T_a, T_0] = 0 \ \forall a$, but appears on the right hand side of some commutators, $[T_a, T_b] = c T_0 + \dots$, for some T_a and T_b , with c being a constant. In the above Virasoro algebra, the role of 0 is played by the term proportional to $\delta_{m,-n}$, which should be viewed as an extra generator in addition to the L_m .

a) As stated in eq. (1), the quantum Virasoro generators are defined as

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathcal{N}(\alpha_{m-n} \cdot \alpha_n), \quad (14)$$

where the normal ordering operator acts as

$$\mathcal{N}(\alpha_m^\mu \alpha_n^\nu) = \begin{cases} \alpha_m^\mu \alpha_n^\nu & \text{for } m \leq n, \\ \alpha_n^\nu \alpha_m^\mu & \text{for } n < m, \end{cases} \quad (15)$$

and the components of the modes α_m , $m \in \mathbb{Z}$ satisfy the commutation relation

$$[\alpha_m^\mu, \alpha_n^\nu] = m \eta^{\mu\nu} \delta_{m,-n}, \quad (16)$$

i.e. the order of α_m^μ and α_n^ν only matters if $m = -n$. Looking at eq. (14), we see that this scenario can only arise if $m = 0$. For $m \neq 0$, $m - n$ can never equal $-n$. Therefore, we only need to worry about normal ordering when treating L_0 .

For $m = 0$ and $n > 0$, the dot product in eq. (14) is already in normal order. For $m = 0$ and $n \leq 0$, the order is reversed. Since we just established that in all other L_m , $m \neq 0$, the order is arbitrary, we can suppress the normal ordering symbol altogether by rewriting

$$\begin{aligned} L_m &= \frac{1}{2} \sum_{n=-\infty}^0 \mathcal{N}(\alpha_{m-n} \cdot \alpha_n) + \frac{1}{2} \sum_{n=1}^{\infty} \mathcal{N}(\alpha_{m-n} \cdot \alpha_n) \\ &= \frac{1}{2} \sum_{n=-\infty}^0 \alpha_n \cdot \alpha_{m-n} + \frac{1}{2} \sum_{n=1}^{\infty} \alpha_{m-n} \cdot \alpha_n. \end{aligned} \quad (17)$$

b) We show by direct calculation that $[\alpha_m^\mu, L_n] = m \alpha_{m+n}^\mu \quad \forall m, n \in \mathbb{Z}$.

$$\begin{aligned} [\alpha_m^\mu, L_n] &= \frac{1}{2} \sum_{l=-\infty}^0 [\alpha_m^\mu, \alpha_{l,\nu} \alpha_{n-l}^\nu] + \frac{1}{2} \sum_{l=1}^{\infty} [\alpha_m^\mu, \alpha_{n-l}^\nu \alpha_{l,\nu}] \\ &= \frac{1}{2} \sum_{l=-\infty}^0 \left([\alpha_m^\mu, \alpha_{l,\nu}] \alpha_{n-l}^\nu + \alpha_{l,\nu} [\alpha_m^\mu, \alpha_{n-l}^\nu] \right) + \frac{1}{2} \sum_{l=1}^{\infty} \left([\alpha_m^\mu, \alpha_{n-l}^\nu] \alpha_{l,\nu} + \alpha_{n-l}^\nu [\alpha_m^\mu, \alpha_{l,\nu}] \right) \\ &\stackrel{(16)}{=} \frac{m}{2} \sum_{l=-\infty}^0 \left(\eta_{\nu}^{\mu} \delta_{m,-l} \alpha_{n-l}^\nu + \eta^{\mu\nu} \delta_{m,l-n} \alpha_{l,\nu} \right) + \frac{m}{2} \sum_{l=1}^{\infty} \left(\eta^{\mu\nu} \delta_{m,l-n} \alpha_{l,\nu} + \eta_{\nu}^{\mu} \delta_{m,-l} \alpha_{n-l}^\nu \right). \end{aligned} \quad (18)$$

We now carry out the above sums over l . When doing so, we have to bear in mind, however, that two of the four Kronecker deltas never contribute. Which ones depends on the values of m and n . E.g. take $m > n > 0$, then the first $\delta_{m,-l}$ contributes at $l = -m$ but $\delta_{m,l-n} = 1 \Leftrightarrow m + n = l$ is always zero since $m + n > 0$. In the second we sum over positive l , so this behavior is reversed. What we end up with is

$$[\alpha_m^\mu, L_n] = \frac{m}{2} \alpha_{m+n}^\mu + \frac{m}{2} \alpha_{m+n}^\mu = m \alpha_{m+n}^\mu. \quad (19)$$

c) Next we demonstrate eq. (6) $\forall m, n \in \mathbb{Z}$. Using $[XY, Z] = X[Y, Z] + [X, Z]Y$, we can write

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2} \sum_{l=-\infty}^0 [\alpha_{l,\mu} \alpha_{m-l}^\mu, L_n] + \frac{1}{2} \sum_{l=1}^{\infty} [\alpha_{m-l}^\mu \alpha_{l,\mu}, L_n] \\
&= \frac{1}{2} \sum_{l=-\infty}^0 \left(\alpha_{l,\mu} [\alpha_{m-l}^\mu, L_n] + [\alpha_{l,\mu}, L_n] \alpha_{m-l}^\mu \right) \\
&\quad + \frac{1}{2} \sum_{l=1}^{\infty} \left(\alpha_{m-l}^\mu [\alpha_{l,\mu}, L_n] + [\alpha_{m-l}^\mu, L_n] \alpha_{l,\mu} \right) \\
&\stackrel{(19)}{=} \frac{1}{2} \sum_{l=-\infty}^0 \left((m-l) \alpha_l \cdot \alpha_{m+n-l} + l \alpha_{n+l} \cdot \alpha_{m-l} \right) \\
&\quad + \frac{1}{2} \sum_{l=1}^{\infty} \left((m-l) \alpha_{m+n-l} \cdot \alpha_l + l \alpha_{m-l} \cdot \alpha_{n+l} \right).
\end{aligned} \tag{20}$$

d) Replacing $p = n + l$ in terms two and four of eq. (20) evidently gives

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2} \sum_{l=-\infty}^0 (m-l) \alpha_l \cdot \alpha_{m+n-l} + \frac{1}{2} \sum_{p=-\infty}^n (p-n) \alpha_p \cdot \alpha_{m+n-p} \\
&\quad + \frac{1}{2} \sum_{l=1}^{\infty} (m-l) \alpha_{m+n-l} \cdot \alpha_l + \frac{1}{2} \sum_{p=n+1}^{\infty} (p-n) \alpha_{m+n-p} \cdot \alpha_p.
\end{aligned} \tag{21}$$

e) Restricting to $n > 0$ and relabeling $l \rightarrow p$ in the first and third term, we may write $[L_m, L_n]$ as

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2} \sum_{p=-\infty}^0 (m-p) \alpha_p \cdot \alpha_{m+n-p} + \frac{1}{2} \sum_{p=-\infty}^n (p-n) \alpha_p \cdot \alpha_{m+n-p} \\
&\quad + \frac{1}{2} \sum_{p=1}^{\infty} (m-p) \alpha_{m+n-p} \cdot \alpha_p + \frac{1}{2} \sum_{p=n+1}^{\infty} (p-n) \alpha_{m+n-p} \cdot \alpha_p.
\end{aligned} \tag{22}$$

We can now partially consolidate sums one and two and sums three and four by splitting sum two at $p = 0$ and sum three at $p = n$ (both operations are legal because $n > 0$) to get

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2} \sum_{p=-\infty}^0 (m-n) \alpha_p \cdot \alpha_{m+n-p} + \frac{1}{2} \sum_{p=1}^n (p-n) \alpha_p \cdot \alpha_{m+n-p} \\
&\quad + \frac{1}{2} \sum_{p=n+1}^{\infty} (m-n) \alpha_{m+n-p} \cdot \alpha_p + \frac{1}{2} \sum_{p=1}^n (m-p) \alpha_{m+n-p} \cdot \alpha_p.
\end{aligned} \tag{23}$$

Assuming further $n > m > 0$, we check for normal ordering in each of the four sums:

sum	p -range	critical value	operator product	mode indices	normal ordered
1	$[-\infty, 0]$	$p = 0$	$\alpha_p \cdot \alpha_{m+n-p}$	$0 < m+n$	✓
2	$[1, n]$	$p = n$	$\alpha_p \cdot \alpha_{m+n-p}$	$n \not< m$	×
3	$[n+1, \infty]$	$p = n+1$	$\alpha_{m+n-p} \cdot \alpha_p$	$m+1 < n+1$	✓
4	$[1, n]$	$p = 1$	$\alpha_{m+n-p} \cdot \alpha_p$	$m+n-1 \not< 1$	×

f) By an application of the modes' commutation relation, we find

$$\begin{aligned}
\sum_{p=1}^n (p-n) \alpha_p \cdot \alpha_{m+n-p} &= \sum_{p=1}^n (p-n) \left(\alpha_{m+n-p} \cdot \alpha_p + \underbrace{[\alpha_{p,\mu}, \alpha_{m+n-p}^\mu]}_{p \eta_\mu^\mu \delta_{p,p-m-n} = p D \delta_{m,-n}} \right) \\
&= \sum_{p=1}^n (p-n) \alpha_{m+n-p} \cdot \alpha_p + \sum_{p=1}^n (p-n) p D \delta_{m,-n}.
\end{aligned} \tag{24}$$

g) By inserting eq. (24) for the second term in eq. (23), we get

$$\begin{aligned}
[L_m, L_n] = & \frac{1}{2} \sum_{p=-\infty}^0 (m-n) \alpha_p \cdot \alpha_{m+n-p} + \frac{1}{2} \sum_{p=1}^n (p-n) \alpha_{m+n-p} \cdot \alpha_p + \frac{1}{2} \sum_{p=1}^n (p-n) p D \delta_{m,-n} \\
& + \frac{1}{2} \sum_{p=n+1}^{\infty} (m-n) \alpha_{m+n-p} \cdot \alpha_p + \frac{1}{2} \sum_{p=1}^n (m-p) \alpha_{m+n-p} \cdot \alpha_p
\end{aligned} \tag{25}$$

where sums two and five combine to give

$$\begin{aligned}
[L_m, L_n] = & \frac{1}{2} \sum_{p=-\infty}^0 (m-n) \alpha_p \cdot \alpha_{m+n-p} + \frac{1}{2} \sum_{p=1}^n (m-n) \alpha_{m+n-p} \cdot \alpha_p \\
& + \frac{1}{2} \sum_{p=1}^n (p-n) p D \delta_{m,-n} + \frac{1}{2} \sum_{p=n+1}^{\infty} (m-n) \alpha_{m+n-p} \cdot \alpha_p.
\end{aligned} \tag{26}$$

By further joining sums two and four and reinserting the normal ordering operator, we obtain

$$\begin{aligned}
[L_m, L_n] = & \frac{1}{2} \sum_{p=-\infty}^0 (m-n) \alpha_p \cdot \alpha_{m+n-p} + \frac{1}{2} \sum_{p=1}^{\infty} (m-n) \alpha_{m+n-p} \cdot \alpha_p + \frac{D}{2} \sum_{p=1}^n p(p-n) \delta_{m,-n} \\
= & \frac{1}{2} \sum_{p=-\infty}^{\infty} (m-n) \mathcal{N}(\alpha_p \cdot \alpha_{m+n-p}) + \frac{D}{2} \sum_{p=1}^n p(p-n) \delta_{m,-n}.
\end{aligned} \tag{27}$$

h) We use induction to proof the following two identities.

$$1. \sum_{q=1}^n q^2 = \frac{n}{6}(n+1)(2n+1) \quad \forall n \in \mathbb{N}:$$

Checking $n = 1$:

$$\sum_{q=1}^1 q^2 = 1 = \frac{1}{6}(1+1)(2+1). \quad \checkmark \tag{28}$$

Checking $n \Rightarrow n+1$:

$$\begin{aligned}
\sum_{q=1}^{n+1} q^2 &= \sum_{q=1}^n q^2 + (n+1)^2 = \frac{n}{6}(n+1)(2n+1) + (n+1)^2 \\
&= \frac{n+1}{6}[n(2n+1) + 6(n+1)] = \frac{n+1}{6}[2n^2 + 7n + 6] \\
&= \frac{n+1}{6}(2n+3)(n+2) = \frac{n+1}{6}[2(n+1) + 1][(n+1) + 1]. \quad \checkmark
\end{aligned} \tag{29}$$

$$2. \sum_{q=1}^n q = \frac{n}{2}(n+1) \quad \forall n \in \mathbb{N}:$$

Checking $n = 1$:

$$\sum_{q=1}^1 q = 1 = \frac{1}{2}(1+1). \quad \checkmark \tag{30}$$

Checking $n \Rightarrow n+1$:

$$\begin{aligned}
\sum_{q=1}^{n+1} q &= \sum_{q=1}^n q + n+1 = \frac{n}{2}(n+1) + n+1 \\
&= \frac{n+1}{2}(n+2) = \frac{n+1}{2}[(n+1) + 1]. \quad \checkmark
\end{aligned} \tag{31}$$

Applying these identities to the sum in the last term in eq. (27), we can simplify,

$$\begin{aligned}\sum_{p=1}^n (p^2 - np) &= \frac{n}{6}(n+1)(2n+1) - n \frac{n}{2}(n+1) = \frac{n}{6}(n+1)[2n+1-3n] \\ &= -\frac{n}{6}(n+1)(n-1) = -\frac{n}{6}(n^2-1).\end{aligned}\tag{32}$$

Inserting this result into eq. (27), we arrive at the commutator of two quantum Virasoro generators,

$$\begin{aligned}[L_m, L_n] &= (m-n)L_{m+n} - \frac{D}{2} \frac{n}{6}(n^2-1) \delta_{m,-n} \\ &= (m-n)L_{m+n} + \frac{D}{12} m(m^2-1) \delta_{m,-n}.\end{aligned}\tag{33}$$

2 The second excited level ghost

Compute

$$\langle \phi_2 | \phi_2 \rangle = \frac{2c_1^2}{25}(D-1)(26-D)\tag{34}$$

for

$$|\phi_2\rangle = \left(c_1 \alpha_{-1}^2 + c_2 \mathbf{p} \cdot \boldsymbol{\alpha}_{-2} + c_3 (\mathbf{p} \cdot \boldsymbol{\alpha}_{-1})^2 \right) |0, p\rangle.\tag{35}$$

Hint: Given $(L_0 - 1)|\phi_2\rangle = L_1|\phi_2\rangle = L_2|\phi_2\rangle = 0$ (setting $a = 1$), determine the relation between c_1 , c_2 and c_3 defining $|\phi_2\rangle$. Then compute $\langle \phi_2 | \phi_2 \rangle$.

Our goal in this exercise is to evaluate the Virasoro constraints¹

$$(L_m - a \delta_{m,0})|\phi\rangle = 0 \quad \forall m \geq 0 \text{ and } \forall |\phi\rangle \in \mathcal{H}_{\text{phys}},\tag{36}$$

to deduce that $D \leq 26$ is a necessary (but not sufficient) condition for a ghost-free theory. To arrive at this conclusion, it is enough to consider the specific state given in eq. (35).² By assuming $|\phi_2\rangle$ to constitute a physical state and setting the normal ordering constant $a = 1$, we get exactly the constraints mentioned in the hint. To unravel, what implications these constraints hold for the allowed values of D and the c_i , $i \in \{1, 2, 3\}$, our strategy will be to commute the Virasoro generators through all of the creation operators in $|\phi_2\rangle$ and let them act directly on $|0, p\rangle$.

First, using $XY = YX + [X, Y]$ on $L_m|\phi_2\rangle$, we get

$$\begin{aligned}L_m|\phi_2\rangle &= \left(c_1 \alpha_{-1}^2 L_m + c_1 [L_m, \alpha_{-1}^2] + c_2 \mathbf{p} \cdot \boldsymbol{\alpha}_{-2} L_m + c_2 [L_m, \mathbf{p} \cdot \boldsymbol{\alpha}_{-2}] \right. \\ &\quad \left. + c_3 (\mathbf{p} \cdot \boldsymbol{\alpha}_{-1})^2 L_m + c_3 [L_m, (\mathbf{p} \cdot \boldsymbol{\alpha}_{-1})^2] \right) |0, p\rangle \\ &= \frac{1}{2} \delta_{m,0} \alpha_0^2 |\phi_2\rangle + \left(c_1 [L_m, \alpha_{-1}^2] + c_2 \mathbf{p} \cdot [L_m, \boldsymbol{\alpha}_{-2}] + c_3 [L_m, (\mathbf{p} \cdot \boldsymbol{\alpha}_{-1})^2] \right) |0, p\rangle.\end{aligned}\tag{38}$$

¹These constraints originated all the way back from $T_{ab} = 0$, which arose as the e.o.m. of the worldsheet metric $h_{ab} = 0$.

Since the Virasoro generators are nothing but the Fourier modes of the energy-momentum tensor, the constraint of having to vanish passes directly on to them.

²As opposed to the most general state $|\phi_2\rangle$ at second excited level which can be written i.t.o. the string field modes as

$$|\phi_2\rangle = \left(\zeta_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu + \eta_\mu \alpha_{-2}^\mu \right) |0, p\rangle.\tag{37}$$

where in the last step, we used that L_m for $m \geq 0$ acts on the vacuum as

$$\begin{aligned}
L_m|0, p\rangle &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathcal{N}(\alpha_{m-n} \cdot \alpha_n) |0, p\rangle \\
&= \frac{1}{2} \sum_{n > \lceil \frac{m}{2} \rceil} \alpha_{m-n} \cdot \underbrace{\alpha_n}_{0} |0, p\rangle + \frac{1}{2} \sum_{n=\lfloor \frac{m}{2} \rfloor}^{\lceil \frac{m}{2} \rceil} \mathcal{N}(\alpha_{m-n} \cdot \alpha_n) |0, p\rangle + \frac{1}{2} \sum_{n < \lfloor \frac{m}{2} \rfloor} \alpha_n \cdot \underbrace{\alpha_{m-n}}_0 |0, p\rangle \\
&= \begin{cases} \frac{1}{2} \alpha_0 \cdot \alpha_0 |0, p\rangle & \text{if } m = 0, \\ \frac{1}{2} \alpha_{m-\frac{m}{2}} \cdot \alpha_{\frac{m}{2}} |0, p\rangle = 0 & \text{if } m \in 2\mathbb{N}, \\ \frac{1}{2} \left(\alpha_{m-\lceil \frac{m}{2} \rceil} \cdot \alpha_{\lceil \frac{m}{2} \rceil} + \alpha_{\lfloor \frac{m}{2} \rfloor} \cdot \alpha_{m-\lfloor \frac{m}{2} \rfloor} \right) |0, p\rangle = 0 & \text{if } m \in 2\mathbb{N} + 1, \end{cases} \\
&= \frac{1}{2} \delta_{m,0} \alpha_0^2 |0, p\rangle
\end{aligned} \tag{39}$$

Recall that modes α_m^μ with $m \leq 0$ were chosen as the creation operators whereas $m > 0$ corresponded to annihilators.

We will now calculate each commutator the last line of eq. (38) in turn by reusing the identities

$$[\alpha_m^\mu, \alpha_n^\nu] = m \eta^{\mu\nu} \delta_{m,-n} \quad \text{and} \quad [\alpha_m^\mu, L_n] = m \alpha_{m+n}^\mu, \tag{40}$$

which have already proven useful in exercise 1.

Using $[X, YZ] = [X, Y]Z + Y[X, Z]$, the first commutator becomes

$$\begin{aligned}
[L_m, \alpha_{-1}^2] &= \underbrace{[L_m, \alpha_{-1}]}_{-(-1)\alpha_{-1+m}} \cdot \alpha_{-1} + \alpha_{-1} \cdot \underbrace{[L_m, \alpha_{-1}]}_{-(-1)\alpha_{-1+m}} = \alpha_{m-1} \cdot \alpha_{-1} + \alpha_{-1} \cdot \alpha_{m-1} \\
&= 2 \alpha_{m-1} \cdot \alpha_{-1} + \underbrace{[\alpha_{-1,\mu}, \alpha_{m-1}^\mu]}_{-\eta^\mu{}_\mu \delta_{-1,1-m}} = 2 \alpha_{m-1} \cdot \alpha_{-1} - D \delta_{m,2}.
\end{aligned} \tag{41}$$

The second commutator is simpler,

$$[L_m, \alpha_{-2}] = -(-2)\alpha_{-2+m} = 2 \alpha_{m-2}, \tag{42}$$

whereas the third commutator is again a little work,

$$\begin{aligned}
[L_m, (\mathbf{p} \cdot \alpha_{-1})^2] &= [L_m, \mathbf{p} \cdot \alpha_{-1}] \mathbf{p} \cdot \alpha_{-1} + \mathbf{p} \cdot \alpha_{-1} [L_m, \mathbf{p} \cdot \alpha_{-1}] \\
&= \mathbf{p} \cdot [L_m, \alpha_{-1}] \mathbf{p} \cdot \alpha_{-1} + \mathbf{p} \cdot \alpha_{-1} \mathbf{p} \cdot [L_m, \alpha_{-1}] \\
&\stackrel{(40)}{=} \mathbf{p} \cdot \alpha_{m-1} \mathbf{p} \cdot \alpha_{-1} + \mathbf{p} \cdot \alpha_{-1} \mathbf{p} \cdot \alpha_{m-1} \\
&= 2 \mathbf{p} \cdot \alpha_{m-1} \mathbf{p} \cdot \alpha_{-1} + \underbrace{[\mathbf{p} \cdot \alpha_{-1}, \mathbf{p} \cdot \alpha_{m-1}]}_{-p_\mu p_\nu \eta^{\mu\nu} \delta_{-1,1-m}} \\
&= 2 \mathbf{p} \cdot \alpha_{m-1} \mathbf{p} \cdot \alpha_{-1} - \mathbf{p}^2 \delta_{m,2}.
\end{aligned} \tag{43}$$

Reinserting eqs. (41) to (43) into eq. (38), we get (still with $m \geq 0$),

$$\begin{aligned}
L_m|\phi_2\rangle &= \frac{1}{2} \delta_{m,0} \alpha_0^2 |\phi_2\rangle + \left(2c_1 \alpha_{m-1} \cdot \alpha_{-1} - c_1 D \delta_{m,2} + 2c_2 \mathbf{p} \cdot \alpha_{m-2} \right. \\
&\quad \left. + 2c_3 \mathbf{p} \cdot \alpha_{m-1} \mathbf{p} \cdot \alpha_{-1} - c_3 \mathbf{p}^2 \delta_{m,2} \right) |0, p\rangle.
\end{aligned} \tag{44}$$

In particular, considering eq. (44) for $m = 0$ yields

$$\begin{aligned}
L_0|\phi_2\rangle &= \frac{1}{2} \alpha_0^2 |\phi_2\rangle + \left(2c_1 \alpha_{-1} \cdot \alpha_{-1} + 2c_2 \mathbf{p} \cdot \alpha_{-2} + 2c_3 \mathbf{p} \cdot \alpha_{-1} \mathbf{p} \cdot \alpha_{-1} \right) |0, p\rangle \\
&= \left(\frac{1}{2} \alpha_0^2 + 2 \right) |\phi_2\rangle.
\end{aligned} \tag{45}$$

Ergo, the physical state condition $(L_0 - a)|\phi\rangle \stackrel{!}{=} 0$, where $a = 1$, implies

$$(L_0 - 1)|\phi_2\rangle = \left(\frac{1}{2}\alpha_0^2 + 1\right)|\phi_2\rangle \stackrel{!}{=} 0 \quad \Rightarrow \quad \alpha_0^2 = -2. \quad (46)$$

For $m = 1$ eq. (44) reads

$$L_1|\phi_2\rangle = \left(2c_1\alpha_0 \cdot \alpha_{-1} + 2c_2\mathbf{p} \cdot \alpha_{-1} + 2c_3\mathbf{p} \cdot \alpha_0\mathbf{p} \cdot \alpha_{-1}\right)|0, p\rangle. \quad (47)$$

To proceed here, we need to recall an equality derived in [exercise 1.e\) on assignment 3](#), namely $\alpha_0 = \sqrt{2\alpha'}\mathbf{p}$. We therefore get

$$\begin{aligned} L_1|\phi_2\rangle &= \left(2c_1\alpha_0 \cdot \alpha_{-1} + \frac{2c_2}{\sqrt{2\alpha'}}\alpha_0 \cdot \alpha_{-1} + \frac{2c_3}{2\alpha'}\underbrace{\alpha_0 \cdot \alpha_0}_{-2}\alpha_0 \cdot \alpha_{-1}\right)|0, p\rangle \\ &= 2\left(c_1 + \frac{c_2}{\sqrt{2\alpha'}} - \frac{c_3}{\alpha'}\right)\alpha_0 \cdot \alpha_{-1}|0, p\rangle \stackrel{!}{=} 0 \quad \Rightarrow \quad c_1 + \frac{c_2}{\sqrt{2\alpha'}} - \frac{c_3}{\alpha'} = 0. \end{aligned} \quad (48)$$

Lastly, for $m = 2$, eq. (44) becomes

$$L_2|\phi_2\rangle = \left(2c_1\alpha_1 \cdot \alpha_{-1} - c_1D + 2c_2\mathbf{p} \cdot \alpha_0 + 2c_3\mathbf{p} \cdot \alpha_1\mathbf{p} \cdot \alpha_{-1} - c_3\mathbf{p}^2\right)|0, p\rangle. \quad (49)$$

Here, we can apply the mode's commutation relation to let α_1 act directly on the vacuum $|0, p\rangle$.

$$\alpha_1 \cdot \alpha_{-1}|0, p\rangle = \underbrace{\alpha_{-1} \cdot \alpha_1}_{0}|0, p\rangle + \underbrace{[\alpha_{1,\mu}, \alpha_{-1}^\mu]}_{\eta^\mu{}_\mu \delta_{1,-(-1)}}|0, p\rangle = D|0, p\rangle, \quad (50)$$

$$\mathbf{p} \cdot \alpha_1\mathbf{p} \cdot \alpha_{-1}|0, p\rangle = p_\mu p_\nu \underbrace{\alpha_{-1}^\mu}_{0} \underbrace{\alpha_1^\nu}_{\eta^{\mu\nu} \delta_{1,-(-1)}}|0, p\rangle + p_\mu p_\nu \underbrace{[\alpha_{-1}^\mu, \alpha_1^\nu]}_{\eta^{\mu\nu} \delta_{1,-(-1)}}|0, p\rangle = \mathbf{p}^2|0, p\rangle. \quad (51)$$

Reinserting eqs. (50) and (51) into eq. (49) and using $\mathbf{p}^2 = \frac{\alpha_0^2}{2\alpha'} = -\frac{1}{\alpha'}$ gives

$$\begin{aligned} L_2|\phi_2\rangle &= \left(2c_1D - c_1D + 2\sqrt{2\alpha'}c_2\mathbf{p}^2 + 2c_3\mathbf{p}^2 - c_3\mathbf{p}^2\right)|0, p\rangle \stackrel{!}{=} 0 \\ &\Rightarrow Dc_1 - 2\sqrt{2\alpha'}c_2 - \frac{c_3}{\alpha'} = 0. \end{aligned} \quad (52)$$

We now have two equations for the three state coefficients c_i , $i \in \{1, 2, 3\}$. We can use them to express c_2 and c_3 i.t.o. c_1 . By Inserting eq. (48) into eq. (52), we find

$$Dc_1 - \frac{4c_2}{\sqrt{2\alpha'}} - c_1 - \frac{c_2}{\sqrt{2\alpha'}} = c_1(D - 1) - \frac{5c_2}{\sqrt{2\alpha'}} = 0 \quad \Rightarrow \quad c_2 = \frac{\sqrt{2\alpha'}}{5}(D - 1)c_1. \quad (53)$$

Plugging this back into eq. (48) gives

$$c_1 + \frac{1}{5}(D - 1)c_1 - \frac{c_3}{\alpha'} = \frac{1}{5}(D + 4)c_1 - \frac{c_3}{\alpha'} = 0 \quad \Rightarrow \quad c_3 = \frac{D + 4}{5}\alpha'c_1. \quad (54)$$

We are finally in a position to calculate $\langle\phi_2|\phi_2\rangle$ and see what constraints on D we must impose in order to avoid a negative norm of $|\phi_2\rangle$. Note that when expanding the ensuing product of modes, we do not need to consider mixed terms in which $\alpha_{\pm 2}$ appears since it freely commutes with $\alpha_{\pm 1}$ and can thus always act directly on the vacuum. This already reduces the total number of resulting terms from nine to five.

$$\begin{aligned} \langle\phi_2|\phi_2\rangle &= \langle 0, p | \left(c_1\alpha_1^2 + c_2\mathbf{p} \cdot \alpha_2 + c_3(\mathbf{p} \cdot \alpha_1)^2 \right) \left(c_1\alpha_{-1}^2 + c_2\mathbf{p} \cdot \alpha_{-2} + c_3(\mathbf{p} \cdot \alpha_{-1})^2 \right) | 0, p \rangle \\ &= c_1^2 \langle 0, p | \alpha_1^2 \alpha_{-1}^2 | 0, p \rangle + c_1 c_3 \langle 0, p | \alpha_1^2 (\mathbf{p} \cdot \alpha_{-1})^2 | 0, p \rangle + c_2^2 \langle 0, p | \mathbf{p} \cdot \alpha_2 \mathbf{p} \cdot \alpha_{-2} | 0, p \rangle \\ &\quad + c_3 c_1 \langle 0, p | (\mathbf{p} \cdot \alpha_1)^2 \alpha_{-1}^2 | 0, p \rangle + c_3^2 \langle 0, p | (\mathbf{p} \cdot \alpha_1)^2 (\mathbf{p} \cdot \alpha_{-1})^2 | 0, p \rangle. \end{aligned} \quad (55)$$

We'll calculate each of the five contributions in turn.

$$\begin{aligned}
\langle 0, p | \alpha_1^2 \alpha_{-1}^2 | 0, p \rangle &= \langle 0, p | \alpha_{1,\mu} (\alpha_{-1}^\nu \alpha_1^\mu + \eta^{\mu\nu}) \alpha_{-1,\nu} | 0, p \rangle \\
&= \langle 0, p | \alpha_{1,\mu} \alpha_{-1}^\nu (\alpha_{-1,\nu} \alpha_1^\mu + \eta^\mu{}_\nu) | 0, p \rangle + \langle 0, p | \alpha_1 \cdot \alpha_{-1} | 0, p \rangle \\
&= \langle 0, p | \alpha_{1,\mu} \alpha_{-1}^\nu \underbrace{\alpha_{-1,\nu} \alpha_1^\mu}_0 | 0, p \rangle + 2 \langle 0, p | \alpha_1 \cdot \alpha_{-1} | 0, p \rangle \stackrel{(50)}{=} 2D,
\end{aligned} \tag{56}$$

$$\begin{aligned}
\langle 0, p | \alpha_1^2 (\mathbf{p} \cdot \alpha_{-1})^2 | 0, p \rangle &= p_\nu p_\rho \langle 0, p | \alpha_{1,\mu} (\alpha_{-1}^\nu \alpha_1^\mu + \eta^{\mu\nu}) \alpha_{-1}^\rho | 0, p \rangle \\
&= p_\nu p_\rho \langle 0, p | \alpha_{1,\mu} \alpha_{-1}^\nu (\alpha_{-1}^\rho \alpha_1^\mu + \eta^{\mu\rho}) | 0, p \rangle + \langle 0, p | \mathbf{p} \cdot \alpha_1 \mathbf{p} \cdot \alpha_{-1} | 0, p \rangle \\
&= p_\nu p_\rho \langle 0, p | \alpha_{1,\mu} \alpha_{-1}^\nu \alpha_{-1}^\rho \underbrace{\alpha_1^\mu}_0 | 0, p \rangle + 2 \langle 0, p | \mathbf{p} \cdot \alpha_1 \mathbf{p} \cdot \alpha_{-1} | 0, p \rangle \stackrel{(51)}{=} 2\mathbf{p}^2,
\end{aligned} \tag{57}$$

$$\langle 0, p | \mathbf{p} \cdot \alpha_2 \mathbf{p} \cdot \alpha_{-2} | 0, p \rangle = p_\mu p_\nu \langle 0, p | (\alpha_{-2}^\mu + 2\eta^{\mu\nu}) | 0, p \rangle = 2\mathbf{p}^2, \tag{58}$$

$$\langle 0, p | (\mathbf{p} \cdot \alpha_1)^2 \alpha_{-1}^2 | 0, p \rangle = (\langle 0, p | \alpha_1^2 (\mathbf{p} \cdot \alpha_{-1})^2 | 0, p \rangle)^\dagger \stackrel{(57)}{=} (2\mathbf{p}^2)^\dagger = 2\mathbf{p}^2, \tag{59}$$

$$\langle 0, p | (\mathbf{p} \cdot \alpha_1)^2 (\mathbf{p} \cdot \alpha_{-1})^2 | 0, p \rangle = 2\mathbf{p}^4. \tag{60}$$

With eqs. (56) to (60) inserted, eq. (55) reads

$$\begin{aligned}
\langle \phi_2 | \phi_2 \rangle &= 2Dc_1^2 + 2\mathbf{p}^2 c_1 c_3 + 2\mathbf{p}^2 c_2^2 + 2\mathbf{p}^2 c_3 c_1 + 2\mathbf{p}^4 c_3^2 \\
&\stackrel{(53)}{=} 2c_1^2 \left[D + 2\mathbf{p}^2 \frac{D+4}{5} \alpha' + \mathbf{p}^2 \left(\frac{\sqrt{2\alpha'}}{5} (D-1) \right)^2 + \mathbf{p}^4 \left(\frac{D+4}{5} \alpha' \right)^2 \right] \\
&= 2c_1^2 \left[D - \frac{2D+8}{5} - \frac{2}{25} (D-1)^2 + \frac{(D+4)^2}{25} \right] \\
&= \frac{2c_1^2}{25} [25D - 10D - 40 - 2D^2 + 4D - 2 + D^2 + 8D + 16] \\
&= \frac{2c_1^2}{25} [27D - 26 - D^2] = \frac{2c_1^2}{25} (D-1)(26-D) < 0 \quad \text{if } D > 26.
\end{aligned} \tag{61}$$

Thus we find that a theory operating in more than 26 dimensions would suffer from negative norm states, so-called ghosts.

3 Some classical lightcone gauge identities

Show that in target-space lightcone gauge the constraint $T_{ab} = 0$ implies

$$\partial_\pm X^- = \frac{l}{2\pi\alpha' p^+} (\partial_\pm X_\perp)^2. \tag{62}$$

Show that for the open string with NN boundary conditions the X^- oscillators can be solved for in terms of the transverse oscillators as

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'} p^+} \frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^i \alpha_m^i. \tag{63}$$

Note that the sum over oscillators includes $\alpha_0^i = \sqrt{2\alpha'} p^i$.

In [exercise 1.b\)](#) on [assignment 2](#), we varied the Polyakov action w.r.t. the worldsheet metric h^{ab} to obtain the classical equation of motion $T_{ab} = 0$. Moving to lightcone gauge, we found that the constraints $T_{+-} = T_{-+} = 0$ were automatically satisfied due to the requirement of tracelessness imposed on the energy momentum tensor T_{ab} as a direct consequence of Weyl invariance.³ The

³The exact argument here was that $0 \stackrel{!}{=} T_a^a = h_{ab} T^{ab} = h_{+-} T^{+-} + h_{-+} T^{-+} = 2h_{+-} T^{+-} = 2(-\frac{1}{2}) T^{+-} = -T^{+-}$, where we used that $h_{++} = h_{--} = 0$, and that h_{ab} and T_{ab} are both symmetric tensors.

components T_{+-} , T_{-+} were a priori non-vanishing and given by

$$T_{ab} = -\frac{1}{\alpha'} \left[\partial_a X \cdot \partial_b X - \frac{1}{2} h_{ab} h^{cd} \partial_c X \cdot \partial_d X \right] \xrightarrow{h_{\pm\pm}=0} T_{\pm\pm} = -\frac{1}{\alpha'} \partial_{\pm} X \cdot \partial_{\pm} X. \quad (64)$$

However, we cannot gauge away dynamical information such as $T_{ab} = 0$ (only redundancies in our physical description are affected by gauge transformations). So if it was true in the original worldsheet coordinates (τ, σ) , it must remain true in lightcone coordinates (ξ^+, ξ^-) . Thus when working in lightcone gauge, we have to implement as constraints

$$T_{++} = T_{--} = 0. \quad (65)$$

Inserting $\partial_{\pm} = \frac{1}{2}(\partial_{\tau} \pm \partial_{\sigma})$ into eq. (64) and enforcing eq. (65) gives

$$(\partial_{\tau} X \pm \partial_{\sigma} X)^2 = 0. \quad (66)$$

This expression becomes useful when we recall that the (flat) ambient space metric $\eta_{\mu\nu}$ in lightcone gauge is given by

$$\left. \begin{aligned} \eta_{+-} &\equiv \eta_{0,D-1} = -1 = \eta_{D-1,0} \equiv \eta_{-+}, \\ \eta_{ij} &= \delta_{ij}, \quad i, j \in \{1, \dots, D-2\}, \end{aligned} \right\} \text{ i.e. } \boldsymbol{\eta} = \begin{pmatrix} 0 & & & -1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ -1 & & & & 0 \end{pmatrix}, \quad (67)$$

so that the Minkowski product of the string field

$$X^2 = \eta_{\mu\nu} X^{\mu} X^{\nu} = -2X^{+}X^{-} + \sum_{i=1}^{D-2} (X^i)^2 \equiv -2X^{+}X^{-} + X_{\perp}^2, \quad (68)$$

where $X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$. Applying this scheme to eq. (66) yields

$$-2(\partial_{\tau} X \pm \partial_{\sigma} X)^+ (\partial_{\tau} X \pm \partial_{\sigma} X)^- + (\partial_{\tau} X \pm \partial_{\sigma} X)_{\perp}^2 = 0. \quad (69)$$

The next step is where working in lightcone gauge pays off. Lightcone gauge implies that we are already dealing with a flat worldsheet, i.e. $h_{ab} = \eta_{ab}$. However, there is still left a residual reparametrization invariance⁴ generated by the conformal Killing vector fields ϵ_a which fulfill $\nabla_a \epsilon_b + \nabla_b \epsilon_a = h_{ab} \nabla_c \epsilon^c$. We can fix this remaining invariance by transforming into a set of coordinates in which we identify the ambient space dimension X^{+} with the worldsheet's time dimension τ :⁵

$$X^{+} = \frac{2\pi\alpha'}{l} p^{+} \tau + x^{+}. \quad (70)$$

This transformation leaves us with

$$\partial_{\tau} X^{+} = \frac{2\pi\alpha'}{l} p^{+} \quad \text{and} \quad \partial_{\sigma} X^{-} = 0, \quad (71)$$

which we can insert into eq. (69) to get

$$-2 \frac{2\pi\alpha'}{l} p^{+} \underbrace{(\partial_{\tau} X \pm \partial_{\sigma} X)^-}_{2\partial_{\pm} X^{-}} + \underbrace{(\partial_{\tau} X \pm \partial_{\sigma} X)_{\perp}^2}_{4(\partial_{\pm} X_{\perp})^2} = 0 \quad \Rightarrow \quad \partial_{\pm} X^{-} = \frac{l}{2\pi\alpha' p^{+}} (\partial_{\pm} X_{\perp})^2. \quad (72)$$

This is the first identity we were asked to derive.

⁴In the lecture notes, this is often referred to as residual conformal symmetry.

⁵This has several advantages unrelated to our line of thought: By fixing the residual reparametrization invariance, all ghosts and unphysical degrees of freedom are eliminated. The disadvantage is that Lorentz covariance becomes non-manifest, i.e. hard to prove.

For the second one, we recall the open string mode expansion for Neumann boundary conditions at both ends as derived in great detail in [exercise 1.b\) on assignment 3](#),

$$X^\mu(\tau, \sigma) = x_0^\mu + \frac{2\pi\alpha'}{l} p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-\frac{\pi i}{l} n \tau} \cos\left(\frac{\pi}{l} n \sigma\right). \quad (73)$$

Differentiation w.r.t. $\xi^\pm$ yields

$$\begin{aligned} \partial_\pm X^\mu(\tau, \sigma) &= \frac{1}{2}(\partial_\tau X^\mu \pm \partial_\sigma X^\mu) \\ &= \frac{\pi\alpha'}{l} p^\mu + \frac{\pi}{l} \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-\frac{\pi i}{l} n \tau} \cos\left(\frac{\pi}{l} n \sigma\right) \mp i \frac{\pi}{l} \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-\frac{\pi i}{l} n \tau} \sin\left(\frac{\pi}{l} n \sigma\right) \\ &= \frac{\pi\alpha'}{l} p^\mu + \frac{\pi}{l} \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-\frac{\pi i}{l} n \tau} e^{\mp \frac{\pi i}{l} n \sigma} \\ &= \frac{\pi}{l} \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-\frac{\pi i}{l} n \xi^\pm}, \end{aligned} \quad (74)$$

where we used $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$ in the last step to consolidate the $n = 0$ -term back into the sum. This is indeed the original Fourier series we started with in [exercise 1.b\)](#) when deriving the string field's mode expansion and thus a good check for consistency.

Inserting this series into relation (72) between X^μ 's lightcone and orthogonal components, we get

$$\begin{aligned} \partial_\pm X^- &= \frac{\pi}{l} \sqrt{\frac{\alpha'}{2}} \sum_{m \in \mathbb{Z}} \alpha_m^- e^{-\frac{\pi i}{l} m \xi^\pm} \\ &\stackrel{!}{=} \frac{l}{2\pi\alpha' p^+} \frac{\pi^2}{l^2} \frac{\alpha'}{2} \sum_{i=1}^{D-2} \sum_{m \in \mathbb{Z}} \alpha_m^i e^{-\frac{\pi i}{l} m \xi^\pm} \sum_{k \in \mathbb{Z}} \alpha_k^i e^{-\frac{\pi i}{l} k \xi^\pm} = \frac{l}{2\pi\alpha' p^+} (\partial_\pm X_\perp)^2. \end{aligned} \quad (75)$$

By multiplying both sides of the above equation with $e^{-\frac{\pi i}{l} n \xi^\pm}$ and integrating $\xi^\pm$ from $-l$ to l , this simplifies to

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \alpha_m^- \underbrace{\int_{-l}^l d\xi^\pm e^{\frac{\pi i}{l} (n-m) \xi^\pm}}_{2l \delta_{n,m}} &= \frac{l}{2\pi\alpha' p^+} \frac{\pi}{l} \sqrt{\frac{\alpha'}{2}} \sum_{i=1}^{D-2} \sum_{k, m \in \mathbb{Z}} \alpha_k^i \alpha_m^i \underbrace{\int_{-l}^l d\xi^\pm e^{\frac{\pi i}{l} (n-m-k) \xi^\pm}}_{2l \delta_{n-m,k}} \\ \Rightarrow \alpha_n^- &= \frac{1}{\sqrt{2\alpha' p^+}} \frac{1}{2} \sum_{i=1}^{D-2} \sum_{m \in \mathbb{Z}} \alpha_{n-m}^i \alpha_m^i. \end{aligned} \quad (76)$$

Thus for the open string with Neumann boundary conditions, all those lightcone modes α_n^- that were not gauged into oblivion by (70) can be expressed in terms of the transverse modes α_n^i , $i \in \{1, \dots, D-2\}$. From this fact, we can infer that the lightcone dimensions don't actually contain any physical degrees of freedom but are pure gauge material.