16.05.2015

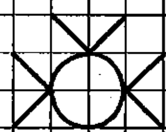
Quantum Field Theory II - Assignment 5

Problem 5.1 (Diagrammatic route to the Coleman-Weinberg potential)

On assignment sheet 4, we looked at how to compute the Coleman-Weinberg potential to one-loop for a ϕ^4-theory using path integral techniques. This week, we compare this result to that computed from only using Feynman diagrams. Recalling that

$$S[\phi] = \int d^4x \left(-\frac{1}{2}\phi(x)\left(\partial_x^2 + \tilde{m}^2\right)\phi(x) - \frac{\lambda}{4!}\phi^4(x)\right). \qquad (1)$$

a) Draw all 1-loop diagrams of ϕ^4-theory (with an arbitrary number of legs), up to order λ^3.

λ:  λ^2:  λ^3: 

b) Show that

$$\Gamma[\phi] \approx S[\phi] - i \ln \int D\phi\, e^{-\frac{1}{2}\int dx\, dy\, \varphi(x) G_0^{-1}(x,y;\phi)\varphi(y)} = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln G_0^{-1}(\phi)$$

with $G_0^{-1}(x,y;\phi) = i\left(\partial_x^2 + m^2 + \frac{\lambda}{2}\phi^2(x)\right)\delta(x-y) \equiv G_0^{-1}(x,y;\phi=0) + i\frac{\lambda}{2}\phi^2(x)\delta(x-y)$.

Hint: You may recall that $\ln(\det A) = \mathrm{Tr}(\ln A)$

To obtain the second equality, we only need to show

$$-i\ln\int D\phi\, e^{-\frac{1}{2}\int dx\, dy\, \varphi(x) G_0^{-1}(x,y;\phi)\varphi(y)} = \frac{i}{2}\mathrm{Tr}\ln\left(G_0^{-1}(\phi)\right)$$

We use the result of exercise 3.2 d) to perform the path integral

$$\int D\phi\, e^{-\frac{1}{2}\int dx\, dy\, \varphi(x) G_0^{-1}(x,y;\phi)\varphi(y)} \equiv \int D\phi\, e^{-\frac{1}{2}\varphi\cdot G_0^{-1}(\phi)\cdot\varphi} = \sqrt{\frac{(2\pi)^n}{\det G_0^{-1}(\phi)}},$$

where $n = \dim(D\phi)$ is the dimensionality of the path integral measure. Reinserting this expression into what we want to show gives

$$-i\ln\sqrt{\frac{(2\pi)^n}{\det G_0^{-1}(\phi)}} = \frac{i}{2}\ln\left(\det G_0^{-1}(\phi)\right) - \frac{i}{2}n\ln(2\pi) = \frac{i}{2}\mathrm{Tr}\ln\left(G_0^{-1}(\phi)\right) - \frac{i}{2}n\ln(2\pi)$$

Apparently, we obtain an extra term of $-\frac{i}{2}n\ln(2\pi)$ and thus cannot confirm eq. (2).

c) Writing $\ln(G_0^{-1}(\phi)) = \ln(G_0^{-1}(0)) + \ln\left(G_0(0)\,G_0^{-1}(\phi)\right)$ with

$$\left(G_0(0)\,G_0^{-1}(\phi)\right)(x,y) = \delta(x-y) + G_0(x,y;0)\,i\frac{\lambda}{2}\phi^2(y), \qquad (5)$$

expand the logarithm and identify your diagrams in part a) with the terms in your expansion.

The Taylor expansion to order n of a function $f(x)$ at point $x = a$ is given by

$$T_n f(x)\big|_{x=a} = \sum_{k=0}^{n} \frac{1}{k!}\frac{d^k f(x)}{d^k x}\bigg|_{x=a}(x-a)^k.$$

The natural logarithm is expanded easiest by writing the argument as $x = 1+y$, yielding

$$\ln(1+y) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}y^n$$

Thus, we find

$$\Gamma[\phi] \approx S[\phi] + \frac{i}{2}\underbrace{\mathrm{Tr}\ln\left(G_0^{-1}(0)\right)}_{\text{needs to vanish, maybe because } = \ln\det[i(\partial^2+m^2)] = 0} + \frac{i}{2}\mathrm{Tr}\ln\left(\delta + G_0(0)\frac{i}{2}\lambda\phi^2\right)$$

$$= S[\phi] + \frac{i}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Tr}\left[G_0(0)\frac{i}{2}\lambda\phi^2\right] =: S[\phi] + I[\phi]$$

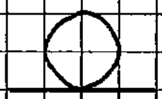
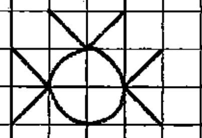
Written out explicitly, $I[\phi]$ reads (due to the trace)

$$I[\phi] = \frac{i}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int dx_1 \ldots \int dx_n\, G_0(x_1,x_2;0)\frac{i}{2}\lambda\phi^2(x_2)\,G_0(x_2,x_3;0)\ldots$$

$$\ldots\frac{i}{2}\lambda\phi^2(x_n)\,G_0(x_n,x_1;0)$$

$$= -\frac{\lambda}{4}\int dx_1\, G_0(x_1, x_1; 0)\, \phi^2(x_1) + \frac{i\lambda^2}{16}\int dx_1 \int dx_2\, G_0(x_1, x_2; 0)\, \phi^2(x_2)\, G_0(x_2, x_1; 0)\, \phi^2(x_1)$$

$$+ \frac{\lambda^3}{48}\int dx_1 \int dx_2 \int dx_3\, G_0(x_1, x_2; 0)\, \phi^2(x_2)\, G_0(x_2, x_3; 0)\, \phi^2(x_3)\, G_0(x_3, x_1; 0)\, \phi^2(x_1) + \dots$$

$\hat{=}$  $+$  $+$  $+ \dots,$

where the last equivalence holds only up to symmetry factors.

d) Verify that this indeed reproduces $V_{eff} = V^{tree} + V^{1\text{-}loop}$ of last week's assignment.

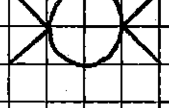
In the referred to problem 4.2, we were considering a constant background field, i.e. $\phi(x_i) = \phi_0 \in \mathbb{C}\ \forall i$.

Inserting for the propagator

$$G(x_1, x_2; 0) = \int \frac{d^4 p_1}{(2\pi)^4} \frac{i}{p_1^2 - m^2 + i\varepsilon} e^{-ip_1(x_1 - x_2)},$$

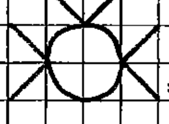
we get

 $= -\frac{\lambda}{4}\int dx_1 \int \frac{dp_1}{(2\pi)^4} \frac{i e^{-ip_1(x_1 - x_1)}}{p_1^2 - m^2 + i\varepsilon}\phi_0^2 = -\frac{\lambda}{4}\phi_0^2\, Vol_{\mathbb{R}^{1,3}} \int \frac{dp_1}{(2\pi)^4} \frac{i}{p_1^2 - m^2 + i\varepsilon},$

 $= \frac{i\lambda^2}{16}\int dx_1 dx_2 \int \frac{d^4 p_1}{(2\pi)^4} \frac{i e^{-ip_1(x_1 - x_2)}}{p_1^2 - m^2 + i\varepsilon}\phi_0^2 \int \frac{d^4 p_2}{(2\pi)^4} \frac{i e^{-ip_2(x_2 - x_1)}}{p_2^2 - m^2 + i\varepsilon}\phi_0^2$

$$= \frac{i\lambda^2}{16}\int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \int dx_1 \int dx_2 \frac{i\phi_0^2}{p_1^2 - m^2 + i\varepsilon} \frac{i\phi_0^2}{p_2^2 - m^2 + i\varepsilon} e^{i(p_2 - p_1)x_1} \underbrace{e^{i(p_1 - p_2)x_2}}_{\int \to (2\pi)^4 \delta(p_1 - p_2)}$$

$$= \frac{i\lambda^2}{16}\phi_0^4\, Vol_{\mathbb{R}^{1,3}} \int \frac{d^4 p_1}{(2\pi)^4} \left(\frac{i}{p_1^2 - m^2 + i\varepsilon}\right)^2,$$

 $= \frac{\lambda^3}{48}\phi_0^6\, Vol_{\mathbb{R}^{1,3}} \int \frac{d^4 p_1}{(2\pi)^4} \left(\frac{i}{p_1^2 - m^2 + i\varepsilon}\right)^3.$

Thus, reintroducing the infinite sum, we obtain

$$\Gamma[\phi] \approx S[\phi] + \frac{i}{2} Vol_{\mathbb{R}^{1,3}} \int \frac{d^4 p}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{\lambda}{2}\phi_0^2 \frac{1}{-p^2 + m^2 + i\varepsilon}\right)^n$$

$$= S[\Phi] + \frac{i}{2} \mathrm{Vol}_{\mathbb{R}^{1,3}} \int \frac{d^4p}{(2\pi)^4} \ln \Big(\underbrace{1 + \frac{\lambda}{2} \phi_0^2 \frac{1}{m^2 - p^2 - i\varepsilon}}_{\frac{m^2 - p^2 - i\varepsilon + \frac{\lambda}{2}\phi_0^2}{m^2 - p^2 - i\varepsilon}} \Big).$$

The above is an intermediate result of problem 4.2 from right before the application of the residue theorem. We may proceed here in the same manner to reobtain our previous result of

$$V_{eff}(\varphi_0) := -\frac{1}{\mathrm{Vol}_{\mathbb{R}^{1,3}}} \Gamma[\varphi_0] \approx V^{tree}(\varphi_0) + V^{1\text{-loop}}(\varphi_0)$$

$$= V^{tree}(\varphi_0) + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{\vec{k}^2 + m^2 + \frac{\lambda}{2}\phi_0^2} - \sqrt{\vec{k}^2 + m^2}$$