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Quantum Field Theory II - Assignment 6

Problem 6.1 (Path integral of the fermionic field with sources)

We define the generating functional for the correlation functions of the free Dirac theory as

$$Z_0[\bar{\eta},\eta] = \int D\bar{\Psi} D\Psi\, e^{i(S_0[\bar{\Psi},\Psi] + \bar{\eta}\cdot\Psi + \bar{\Psi}\eta)} \qquad (1)$$

with $S_0[\bar{\Psi},\Psi] = \int d^4x\, \bar{\Psi}(i\not{\partial} - m_0)\Psi$

a) Show by means of completing the square that (1) can be rearranged to

$$Z_0[\bar{\eta},\eta] = Z_0[\bar{\eta}=0,\eta=0]\, e^{-\bar{\eta}\cdot S_F\cdot\eta}, \qquad (2)$$

where S_F is the Feynman propagator of the free Dirac theory,

$$(i\not{\partial}_x - m_0 + i\epsilon) S_F(x-y) = i\delta^{(4)}(x-y).$$

Since we are working in a free theory, we may choose to shift not the time into the complex plane via $t \to t(1-i\epsilon)$ but instead make the replacement

$$\mathcal{H} \to \mathcal{H} - i\epsilon\,\phi^2(x), \quad \text{i.e. } m_0^2 \to m_0^2 - i\epsilon, \quad \text{(see eq. 7.80)}$$

in order to project the initial and final states to the vacuum. Thus, we may write the action as

$$S_0[\bar{\Psi},\Psi] \to \int d^4x \int d^4y\, \bar{\Psi}\underbrace{(i\not{\partial} - m_0 + i\epsilon)\delta(x-y)}_{iS_F^{-1}(x-y)}\Psi = i\bar{\Psi}\cdot S_F^{-1}\cdot\Psi.$$

Completing the square yields

$$S_0[\bar{\Psi},\Psi] + \bar{\eta}\cdot\Psi + \bar{\Psi}\cdot\eta = i\bar{\Psi}\cdot S_F^{-1}\cdot\Psi + \bar{\eta}\cdot\Psi + \bar{\Psi}\cdot\eta$$

$$= (\bar{\Psi} - iS_F\bar{\eta})\cdot iS_F^{-1}\cdot(\Psi - iS_F\eta) + i\bar{\eta}\cdot S_F\cdot\eta$$

We define $\overline{\Psi}' = \overline{\Psi} - i S_F \overline{\eta}$ and $\Psi' = \Psi + i S_F \eta$ and transform our path integration to these new variables. Since we performed just a shift, the Jacobian of the transformation is in both cases just one. Thus, the integration measure remains invariant. Altogether, this gives

$$Z_0[\overline{\eta}, \eta] = \int \mathcal{D}\overline{\Psi}' \mathcal{D}\Psi' \, e^{i(\underbrace{\overline{\Psi}' \cdot i S_F^{-1} \cdot \Psi'}_{S_0[\overline{\Psi}', \Psi']} + i \overline{\eta} \cdot S_F \cdot \eta)}$$

$$= Z_0[\overline{\eta} = 0, \eta = 0] \, e^{-\overline{\eta} \cdot S_F \cdot \eta}$$

b) We add now an interaction part $\int d^4x \, \mathcal{L}_{int}(\overline{\Psi}, \Psi)$ to $S_0[\overline{\Psi}, \Psi]$. By proceeding as in the lecture for the bosonic case, show that the generating functional for the interacting theory can be written as

$$Z[\overline{\eta}, \eta] = Z_0[\overline{\eta} = 0, \eta = 0] \, e^{-\frac{\delta}{\delta \Psi} S_F \cdot \frac{\delta}{\delta \overline{\Psi}}} \, e^{i \int d^4x \, \mathcal{L}_{int}(\overline{\Psi}, \Psi)}$$

$$\cdot \, e^{i \overline{\eta} \Psi + i \overline{\Psi} \cdot \eta} \Big|_{\overline{\Psi} = \Psi = 0}. \qquad (3)$$

Our full action now reads

$$S[\overline{\Psi}, \Psi] = S_0[\overline{\Psi}, \Psi] + S_{int}[\overline{\Psi}, \Psi],$$

which makes our generating functional

$$Z[\overline{\eta}, \eta] := \int \mathcal{D}\overline{\Psi} \mathcal{D}\Psi \, e^{i(S[\overline{\Psi}, \Psi] + \overline{\eta} \cdot \Psi + \overline{\Psi} \cdot \eta)}$$

$$= \int \mathcal{D}\overline{\Psi} \mathcal{D}\Psi \, e^{i \int d^4x \, \mathcal{L}_{int}(\overline{\Psi}, \Psi)} \, e^{i(S_0[\overline{\Psi}, \Psi] + \overline{\eta} \cdot \Psi + \overline{\Psi} \cdot \eta)}$$

$$= e^{i \int d^4x \, \mathcal{L}_{int}(\frac{-\delta}{i \delta \eta}, \frac{\delta}{i \delta \overline{\eta}})} \int \mathcal{D}\overline{\Psi} \mathcal{D}\Psi \, e^{i(S_0[\overline{\Psi}, \Psi] + \overline{\eta} \cdot \Psi + \overline{\Psi} \cdot \eta)},$$

where we used $F[\Psi] \, e^{i \overline{\eta} \cdot \Psi} = F[\frac{\delta}{i \delta \overline{\eta}}] \, e^{i \overline{\eta} \cdot \Psi}$ for any functional $F[\Psi]$.

Inserting our result from part a), we obtain

$$Z[\bar{\eta},\eta] = e^{i\int d^4x\, \mathcal{L}_{int}\left(\frac{\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}\right)} Z_0[\bar{\eta},\eta]$$

$$= Z_0[\bar{\eta}=0, \eta=0]\, e^{i\int d^4x\, \mathcal{L}_{int}\left(\frac{-\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}\right)} e^{i\bar{\eta}\cdot S_F \cdot i\eta}$$

To proceed, we need to understand how to commute two Grassmann valued functionals containing derivatives. Note that

$$F\left[\frac{\delta}{i\delta\bar{\eta}}\right] G[\bar{\eta}]\, e^{i\bar{\eta}\cdot\psi} = F\left[\frac{\delta}{i\delta\bar{\eta}}\right] G\left[\frac{-\delta}{i\delta\psi}\right] e^{i\bar{\eta}\cdot\psi}$$

$$= (-1)^{\deg(F)\deg(G)}\, G\left[-\frac{\delta}{i\delta\psi}\right] F\left[\frac{\delta}{i\delta\bar{\eta}}\right] e^{i\bar{\eta}\cdot\psi} = (-1)^{\deg(F)\deg(G)}\, G\left[-\frac{\delta}{i\delta\psi}\right] F[\psi]\, e^{i\bar{\eta}\cdot\psi},$$

where we used eq. (7.239) to introduce a factor of $(-1)^{\deg(F)\deg(G)}$ with $\deg(F), \deg(G) \in \{0,1\}$, depending on whether F and G are Grassmann even or odd, also called bosonic and fermionic, respectively. Since we are treating fermions here, $\psi, \bar{\psi}, \eta$, and $\bar{\eta}$ are all Grassmann odd. However, that makes both of the exponentials in the above expression for $Z[\bar{\eta},\eta]$ Grassmann even. Thus

$$Z[\bar{\eta},\eta] = Z_0[\bar{\eta}=0,\eta=0]\, e^{i\int d^4x\, \mathcal{L}_{int}\left(\frac{-\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}\right)} e^{i\bar{\eta}\cdot S_F\cdot i\eta}\, \underbrace{e^{i(\bar{\eta}\cdot\psi+\bar{\psi}\cdot\eta)}\Big|_{\bar{\psi}=\psi=0}}_{1}$$

$$= Z_0[\bar{\eta}=0, \eta=0]\, e^{\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar{\psi}}} e^{i\int d^4x\, \mathcal{L}_{int}(\bar{\psi},\psi)} e^{i(\bar{\eta}\cdot\psi+\bar{\psi}\cdot\eta)}\Big|_{\bar{\psi}=\psi=0}$$

Problem 6.2 (Wick's theorem for fermions)

From eq. (3) one may derive that the correlation function for the free theory are given by

$$\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \bar{\Psi}(x_j) \rangle = e^{\frac{\delta}{\delta\Psi} S_F \cdot \frac{\delta}{\delta\bar{\Psi}}} \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \bar{\Psi}(x_j) \Big|_{\bar{\Psi}=\Psi=0} \quad (5)$$

a) Prove Wick's theorem for the free fermions, i.e. show that $\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \bar{\Psi}(x_j) \rangle = 0$ for $n \neq m$ and

$$\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \bar{\Psi}(x_j) \rangle = (-1)^{\frac{n}{2}(n-1)} S_F(x_1 - x_{n+1}) S_F(x_2 - x_{n+2}) \dots S_F(x_n - x_{2n})$$

+ 'all other contractions with appropriate signs', (6)

for $n = m$.

We can get a better grip on the correlation function in eq. (5) by writing the exponential as a series expansion.

$$\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \bar{\Psi}(x_j) \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{\delta}{\delta\Psi} \cdot S_F \cdot \frac{\delta}{\delta\bar{\Psi}} \right)^k \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \bar{\Psi}(x_j) \Big|_{\bar{\Psi}=\Psi=0}$$

$$= \sum_{k=0}^{\min\{m,n\}} \frac{1}{k!} \left(-\frac{\delta}{\delta\Psi} \cdot S_F \cdot \frac{\delta}{\delta\bar{\Psi}} \right)^k \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \bar{\Psi}(x_j) \Big|_{\bar{\Psi}=\Psi=0}$$

where the sum aborts at $\min\{m,n\}$ since $m+1$ functional derivatives acting on just m fields always gives zero.

If further $m \neq n$, then after applying all functional derivatives, either Ψ-fields $(n > m)$ or $\bar{\Psi}$-fields $(m > n)$ remain in every term appearing due to the product rule. Taking $\bar{\Psi} = \Psi = 0$ in the end makes the whole correlation function vanish. Physically, this reflects charge conservation.

On the other hand if $n = m$, we get

$$\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{2n} \bar{\Psi}(x_j) \rangle = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{\delta}{\delta\bar{\Psi}} \cdot S_F \cdot \frac{\delta}{\delta\Psi} \right)^k \prod_{i=1}^{n} \Psi(x_i) \bar{\Psi}(x_{n+i}) \Big|_{\bar{\Psi}=\Psi=0}$$

Again due to taking the limit $\bar{\Psi}=\Psi=0$, in the end all terms of the sum over k disappear except for $k=n$, where we have an equal number of fields and derivatives. Thus

$$\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{2n} \bar{\Psi}(x_j) \rangle = \frac{1}{n!}\left(\frac{\delta}{\delta \Psi} S_F \cdot \frac{\delta}{\delta \bar{\Psi}}\right)^n \prod_{i=1}^{n} \Psi(x_i) \bar{\Psi}(x_{n+i})$$

$$= (-1)^{\frac{n}{2}(n-1)} S_F(x_1 - x_{n+1}) S_F(x_2 - x_{n+2}) \dots S_F(x_n - x_{2n})$$

+ 'all other contractions with appriate signs'.

The last step actually requires lots of tedious combinatorics and commuting of Grassmann odd fields and derivatives.

b) Show by means of eq. (6) that

$$\langle T \bar{\Psi}_A(x) M^A{}_B \Psi^B(x) \bar{\Psi}_C(y) \hat{M}^C{}_D \Psi^D(y) \rangle = -\mathrm{Tr}\left(S_F(y-x) M S_F(x-y) \hat{M}\right)$$

+ 'uninteresting' $S_F(0)$-terms.

To apply eq. (6), we note that in this case, $n=2$ and so $(-1)^{\frac{n}{2}(n-1)} = -1$.

$$\langle T \bar{\Psi}_A(x) M^A{}_B \Psi^B(x) \bar{\Psi}_C(y) \hat{M}^C{}_D \Psi^D(y) \rangle$$

$$= (-1)^3 M^A{}_B \hat{M}^C{}_D \langle T \Psi^B(x) \Psi^D(y) \bar{\Psi}_A(x) \bar{\Psi}_C(y) \rangle$$

$$= (-1)^3 M^A{}_B \hat{M}^C{}_D \left(-S^B_{F,A}(x-x) S^D_{F,C}(y-y) + S^B_{F,C}(x-y) S^D_{F,A}(y-x)\right)$$

$$= -S^D_{F,A}(y-x) M^A{}_B S^B_{F,C}(x-y) \hat{M}^C{}_D + S^B_{F,A}(0) M^A{}_B S^D_{F,C} \hat{M}^C{}_D$$

$$= -\mathrm{Tr}\left(S_F(y-x) M S_F(x-y) \hat{M}\right) + \text{'}S_F(0)\text{-terms'}$$