String Theory

Solution to Assignment 7

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1 Primary fields and radial quantization

A primary field $\phi(z, \bar{z})$ is a tensor field under conformal transformations $z \to z'$, $\bar{z} \to \bar{z}'$ in the sense that

$$\phi(z, \bar{z}) \to \phi'(z', \bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}), \tag{1}$$

where z, $\bar{z}$ denote arbitrary holomorphic functions.

a) How does a primary field with conformal weights h, $\bar{h}$ transform under dilations $z \to e^{\lambda} z$ and rotations $z \to e^{i\theta} z$ with $\lambda, \theta \in \mathbb{R}$? Give a physical interpretation for $h + \bar{h}$ and $h - \bar{h}$.

b) Consider the infinitesimal conformal transformation

$$z'(z) = z + \zeta(z), \qquad \bar{z}'(\bar{z}) = \bar{z} + \bar{\zeta}(\bar{z}). \tag{2}$$

Show that the infinitesimal transformation of a primary field is given by

$$\delta_{\zeta,\bar{\zeta}} \phi(z, \bar{z}) = \left[h \partial_z \zeta + \bar{h} \partial_{\bar{z}} \bar{\zeta} + \zeta \partial_z + \bar{\zeta} \partial_{\bar{z}}\right] \phi(z, \bar{z}). \tag{3}$$

c) On the cylinder with coordinates (τ, σ) we defined

$$\xi^{\pm} = \tau \pm \sigma = -i(\tau' \pm i\sigma), \qquad \tau = -i\tau', \tag{4}$$

and τ' is the time in Euclidean signature (after Wick rotation). After relabelling $\tau' \to \tau$ we have

$$\xi^{+} = -i\bar{\omega}, \qquad \xi^{-} = -i\omega, \qquad \omega = \tau - i\sigma. \tag{5}$$

Argue that the conformal map from the cylinder to the Riemann sphere, defined by

$$\omega \to z(\omega) = e^{\frac{2\pi}{l}\omega}, \tag{6}$$

maps left-/right-moving fields to holomorphic/antiholomorphic fields and that for a primary field $\Phi = \Phi_L(\xi^-) + \Phi_R(\xi^+)$ a mode expansion

$$\Phi_L(\xi^-) = \left(\tfrac{2\pi}{l}\right)^h \sum_n \phi_n \, e^{i\frac{2\pi}{l} n \xi^-}, \qquad \Phi_R(\xi^+) = \left(\tfrac{2\pi}{l}\right)^{\bar{h}} \sum_n \tilde{\phi}_n \, e^{i\frac{2\pi}{l} n \xi^+}, \tag{7}$$

translates into $\Phi(z, \bar{z}) = \Phi(z) + \bar{\Phi}(\bar{z})$,

$$\Phi(z) = \sum_n \phi_n \, z^{-n-h}, \qquad \bar{\Phi}(\bar{z}) = \sum_n \phi_n \bar{z}^{-n-\bar{h}}. \tag{8}$$

d) Argue that a time-ordered product of fields on the cylinder maps to a radially ordered product

$$\mathcal{R}[\Phi_1(z_1)\Phi_2(z_2)] = \begin{cases} \Phi_1(z_1)\Phi_2(z_2) & \text{for } |z_1| > |z_2|, \\ \Phi_2(z_2)\Phi_1(z_1) & \text{for } |z_2| > |z_1|. \end{cases} \tag{9}$$

a) Under dilations (rescalings) $z \to z' = e^{\lambda} z$, $\lambda \in \mathbb{R}$, a primary field $\phi(z, \bar{z})$ with conformal weights h, $\bar{h}$ transforms as

$$\phi(z,\bar{z}) \to \phi'(z',\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) = e^{-h\lambda} e^{-\bar{h}\lambda} \phi(z,\bar{z}) = e^{-(h+\bar{h})\lambda}\phi(z,\bar{z}). \tag{10}$$

Under rotations $z \to z' = e^{i\theta} z$, $\theta \in \mathbb{R}$, $\phi(z, \bar{z})$ becomes

$$\phi(z,\bar{z}) \to \phi'(z',\bar{z}') = e^{-ih\theta} e^{i\bar{h}\theta} \phi(z,\bar{z}) = e^{-i(h-\bar{h})\theta}\phi(z,\bar{z}). \tag{11}$$

From eqs. (10) and (11), we infer that the sum of conformal weights $h + \bar{h} = \Delta$ gives the **scaling dimension** of a primary field $\phi(z, \bar{z})$, whereas the difference $h - \bar{h} = s$ is the field's (conformal) **spin**.

b) Equation (1) can equivalently be written as

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z',\bar{z}'). \tag{12}$$

For the conformal transformations $z'(z) = z + \zeta(z)$, $\bar{z}'(\bar{z}) = \bar{z} + \bar{\zeta}(\bar{z})$, we can expand each term in eq. (12) to first order in the infinitesimal (anti-)chiral fields $\zeta(z)$, $\bar{\zeta}(\bar{z})$ to get

$$\left(\frac{\partial z'}{\partial z}\right)^{-h} = \left(1 + \frac{\partial \zeta(z)}{\partial z}\right)^{-h} = 1 + h\partial_z \zeta(z) + \mathcal{O}(\zeta^2), \tag{13}$$

$$\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} = \left(1 + \frac{\partial \bar{\zeta}(\bar{z})}{\partial \bar{z}}\right)^{-\bar{h}} = 1 + \bar{h}\partial_{\bar{z}} \bar{\zeta}(\bar{z}) + \mathcal{O}(\bar{\zeta}^2), \tag{14}$$

$$\phi(z',\bar{z}') = \phi(z,\bar{z}) + \zeta(z)\,\partial_z\phi(z,\bar{z}) + \bar{\zeta}(\bar{z})\,\partial_{\bar{z}}\phi(z,\bar{z}) + \mathcal{O}(\zeta^2,\bar{\zeta}^2). \tag{15}$$

Dropping all terms beyond linear order gives the infinitesimal transformation

$$\begin{aligned} \delta_{\zeta,\bar{\zeta}}\phi(z,\bar{z}) &\equiv \phi'(z,\bar{z}) - \phi(z,\bar{z}) \\ &= \Big(1 + h\,\partial_z\zeta(z)\Big)\Big(1 + \bar{h}\,\partial_{\bar{z}}\bar{\zeta}(\bar{z})\Big)\Big(1 + \zeta(z)\,\partial_z + \bar{\zeta}(\bar{z})\,\partial_{\bar{z}}\Big)\phi(z,\bar{z}) - \phi(z,\bar{z}) \\ &= \Big[1 + h\,\partial_z\zeta(z) + \bar{h}\,\partial_{\bar{z}}\bar{\zeta}(\bar{z}) + \zeta(z)\,\partial_z + \bar{\zeta}(\bar{z})\,\partial_{\bar{z}}\Big]\phi(z,\bar{z}) - \phi(z,\bar{z}) \\ &= \Big[h\,\partial_z\zeta(z) + \bar{h}\,\partial_{\bar{z}}\bar{\zeta}(\bar{z}) + \zeta(z)\,\partial_z + \bar{\zeta}(\bar{z})\,\partial_{\bar{z}}\Big]\,\phi(z,\bar{z}). \end{aligned} \tag{16}$$

c) A Wick rotation on the cylinder with coordinates $(\tau, \sigma) \in \mathbb{R}^2$ sends

$$\xi^{\pm} = \tau \pm \sigma \xrightarrow{\text{W.R.}} -i(\tau \pm i\sigma) \equiv \begin{cases} -i\bar{\omega} = \xi^+, \\ -i\omega = \xi^-, \end{cases} \tag{17}$$

where now $\tau \in i\mathbb{R}$ and ω, $\bar{\omega}$ are coordinates on the cylinder.

After Wick rotating, the conformal map

$$i\xi^- = \omega \to z(\omega) = e^{\frac{2\pi}{l}\omega}, \qquad i\xi^+ = \bar{\omega} \to \bar{z}(\bar{\omega}) = e^{\frac{2\pi}{l}\bar{\omega}}. \tag{18}$$

from the cylinder to the Riemann sphere respectively the complex plane transforms left-movers into holomorphic fields $\Phi_L(\xi^-) \to \Phi(z(\omega))$ ones and right-movers into antiholomorphic ones $\Phi_R(\xi^+) \to \Phi(\bar{z}(\bar{\omega}))$.

This can be illustrated for the case of a primary field $\Phi(\xi^+, \xi^-) = \Phi_L(\xi^-) + \Phi_R(\xi^+)$ with conformal weights $(h, \bar{h})$, where the left- and right-moving components are given by

$$\Phi_L(\xi^-) = \left(\tfrac{2\pi}{l}\right)^h \sum_n \phi_n \, e^{i\frac{2\pi}{l} n\xi^-}, \qquad \Phi_R(\xi^+) = \left(\tfrac{2\pi}{l}\right)^{\bar{h}} \sum_n \tilde{\phi}_n \, e^{i\frac{2\pi}{l} n\xi^+}. \tag{19}$$

Under (18), these transform into

$$\begin{aligned} \Phi_L(\xi^-) \to \Phi_L(z) &= \left(\frac{\partial z}{\partial \omega}\right)^{-h} \Phi_L(\xi^-) = \left(\tfrac{2\pi}{l} \, e^{\frac{2\pi}{l}\omega}\right)^{-h} \left(\tfrac{2\pi}{l}\right)^h \sum_n \phi_n \, e^{-\frac{2\pi}{l} n\omega} \\ &= \sum_n \phi_n \, e^{(\frac{2\pi}{l}\omega)^{-n-h}} = \sum_n \phi_n \, z^{-n-h}, \end{aligned} \tag{20}$$

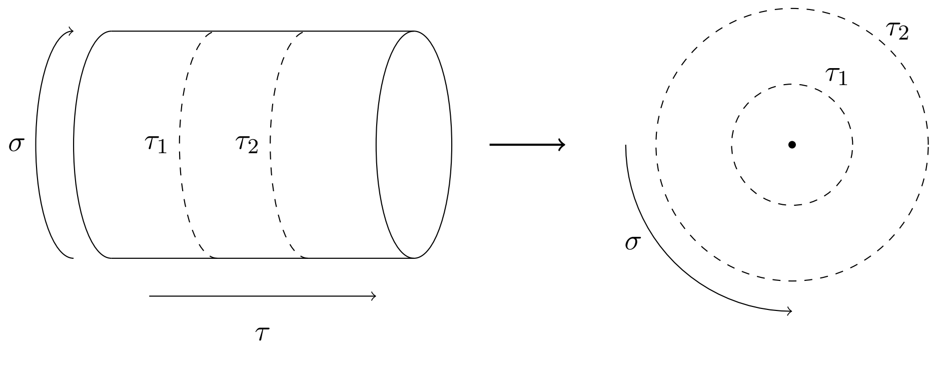
$$\begin{aligned} \Phi_R(\xi^+) \to \Phi_R(\bar{z}) &= \left(\frac{\partial \bar{z}}{\partial \bar{\omega}}\right)^{-\bar{h}} \Phi_R(\xi^+) = \left(\tfrac{2\pi}{l} \, e^{\frac{2\pi}{l}\bar{\omega}}\right)^{-\bar{h}} \left(\tfrac{2\pi}{l}\right)^{\bar{h}} \sum_n \tilde{\phi}_n \, e^{-\frac{2\pi}{l} n\bar{\omega}} \\ &= \sum_n \tilde{\phi}_n \, e^{(\frac{2\pi}{l}\bar{\omega})^{-n-\bar{h}}} = \sum_n \tilde{\phi}_n \, z^{-n-\bar{h}}. \end{aligned} \tag{21}$$

d) To see that time ordering on the cylinder corresponds to radial ordering on the complex plane, we choose any two points $\omega_1 = \tau_1 - i\sigma_1$, $\omega_2 = \tau_2 - i\sigma_2$ such that $\tau_1 \leq \tau_2$, i.e. in a time-ordered operator product containing τ_1 and τ_2, the operator evaluated at τ_1 would act first. For the corresponding points z_1, z_2 on the plane, we have the relation

$$\begin{aligned} \tau_1 \leq \tau_2 \quad \iff \quad |z_1| &= \left| e^{\frac{2\pi}{l}\omega_1} \right| = \left| e^{\frac{2\pi}{l}(\tau_1 - i\sigma_1)} \right| = e^{\frac{2\pi}{l}\tau_1} \\ &\leq e^{\frac{2\pi}{l}\tau_2} = \left| e^{\frac{2\pi}{l}(\tau_2 - i\sigma_2)} \right| = |z_2|. \end{aligned} \tag{22}$$

Thus in a radially ordered product, an operator evaluated on the plane at z_1 would also act first.

The following graphic visualizes how different times on the cylinder correspond to different times on the plane.



2 The OPE of the energy momentum tensor

In the complex plane, the Virasoro generators L_n, $n \in \mathbb{Z}$ are given by

$$L_n = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} \, z^{n+1} \, T(z). \tag{23}$$

a) Show that

$$[L_m, L_n] = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1}\, w^{n+1}\, \mathcal{R}[T(z)\, T(w)], \tag{24}$$

where C_0 denotes a contour about $z = 0$ and C_w is a contour about $z = w$. As usual the product $\mathcal{R}[T(z)\, T(w)]$ is meant to be the radially ordered product.

Hint: Write the commutator as a difference of two double contour integrals and use a contour deformation of the dz integration for fixed w.

b) Use eq. (24) and the (radially ordered) operator product

$$\mathcal{R}[T(z)\, T(w)] = \frac{c/2}{(z-w)^4} + \frac{2\, T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \underbrace{\mathcal{O}[(z-w)^0]}_{\text{finite terms}}, \tag{25}$$

as well as the Cauchy-Riemann formula,

$$\oint_{C_w} \frac{\mathrm{d}z'}{2\pi i} \frac{f(z')}{(z'-z)^n} = \frac{1}{(n-1)!} f^{(n-1)}(z), \tag{26}$$

to rederive the quantum Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n}. \tag{27}$$

c) The Schwarzian derivative is defined as

$$S(\tilde{z}, z) = \frac{\partial^3 \tilde{z}}{\partial^3 z} \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2} \left(\frac{\partial^2 \tilde{z}}{\partial^2 z}\right)^2 \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2}. \tag{28}$$

Show that the transformation

$$T(z) \to \tilde{T}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \left[T(z) - \frac{c}{12} S(\tilde{z}, z)\right], \tag{29}$$

gives at the infinitesimal level for $\tilde{z} = z + \epsilon(z)$,

$$\delta T(z) = \epsilon(z)\partial T(z) + 2[\partial_z \epsilon(z)] T(z) + \frac{c}{12} \partial_z^3 \epsilon(z). \tag{30}$$

d) Apply this to the map from the cylinder to the plane, given by

$$\tau - i\sigma \equiv \omega \to z(\omega) = e^{\frac{2\pi}{l}\omega}, \tag{31}$$

to show that

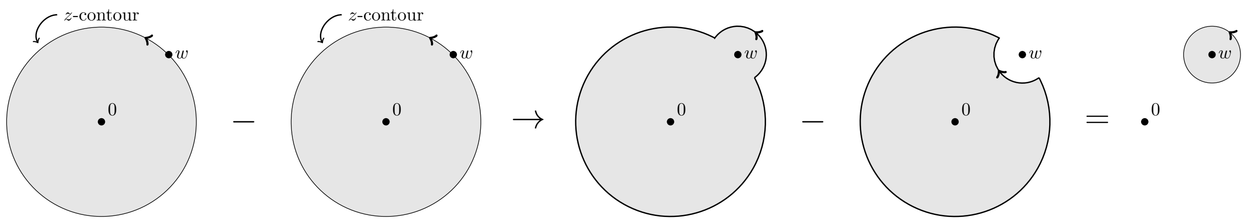
$$T_{\text{cyl}}(\omega) = \left(\tfrac{2\pi}{l}\right)^2 \left(z^2 T_{\text{pln}}(z) - \frac{c}{24}\right). \tag{32}$$

Which are the various physical interpretations for c?

a) Expanding the commutator of Virasoro generators gives

$$\begin{aligned}
[L_m, L_n] &= \oint_{C_0} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1}\, T(z) \oint_{C_0} \frac{\mathrm{d}w}{2\pi i}\, w^{n+1}\, T(w) - \oint_{C_0} \frac{\mathrm{d}w}{2\pi i}\, w^{n+1}\, T(w) \oint_{C_0} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1}\, T(z) \\
&= \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \left(\oint_{C_0} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1}\, w^{n+1}\, T(z)\, T(w) - \oint_{C_0} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1}\, w^{n+1}\, T(w)\, T(z) \right).
\end{aligned} \tag{33}$$

We use the following contour deformation for the two integrals in parenthesis so that *after* the deformation, we have $|w| \leq |z|\ \forall\, z$ in the first integral and $|w| \geq |z|\ \forall\, z$ in the second.



Then eq. (33) simplifies to

$$[L_m, L_n] = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1}\, w^{n+1}\, \mathcal{R}[T(z)\, T(w)]. \tag{34}$$

b) To rederive[1] the quantum Virasoro algebra, we insert eq. (25) into eq. (34) to get

$$[L_m, L_n] = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1}\, w^{n+1} \left[\frac{c/2}{(z-w)^4} + \frac{2\, T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}\right] \tag{35}$$

We can perform the z-integration using the Cauchy-Riemann formula

$$\oint_{C_w} \frac{\mathrm{d}z'}{2\pi i} \frac{f(z')}{(z'-z)^n} = \frac{1}{(n-1)!} f^{(n-1)}(z). \tag{36}$$

The three terms in brackets thus become

$$\oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{\frac{c}{2}\, z^{m+1}\, w^{n+1}}{(z-w)^4} = \frac{c}{2\cdot 3!}(m+1)m(m-1)w^{m-2}w^{n+1} = \frac{c}{12} m(m^2-1) w^{m-n-1}, \tag{37}$$

$$\oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{2\, T(w)\, z^{m+1}\, w^{n+1}}{(z-w)^2} = 2\, T(w)\, (m+1) w^m w^{n+1}, \tag{38}$$

$$\oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{\partial_w T(w)\, z^{m+1}\, w^{n+1}}{z-w} = \partial_w T(w)\, w^{m+1}\, w^{n+1}. \tag{39}$$

We reinsert eqs. (37) to (39) into eq. (35) and get

$$\begin{aligned} [L_m, L_n] &= \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \left[\frac{c}{12} m(m^2-1) w^{m-n-1} + 2\, T(w)\, (m+1) w^m w^{n+1} + \partial_w T(w)\, w^{m+1}\, w^{n+1}\right] \\ &= \frac{c}{12} m(m^2-1)\delta_{m,-n} + \underbrace{\oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \Big[T(w)\, (m+1) w^m w^{n+1}}_{(m+1)\, L_{m+n}} + \partial_w [T(w)\, w^{m+1}]\, w^{n+1}\Big] \end{aligned} \tag{40}$$

Integration by parts in the last term where the boundary terms cancel since we integrate over the closed contour C_0 gives

$$\begin{aligned} \oint_{C_0} \frac{\mathrm{d}w}{2\pi i}\, \partial_w [T(w)\, w^{m+1}]\, w^{n+1} &= -\oint_{C_0} \frac{\mathrm{d}w}{2\pi i}\, T(w)\, w^{m+1}\, \partial_w w^{n+1} \\ &= -(n+1) \oint_{C_0} \frac{\mathrm{d}w}{2\pi i}\, T(w)\, w^{m+1}\, w^n = -(n+1) L_{m+n}. \end{aligned} \tag{41}$$

Thus we arrive at the renowned Virasoro algebra

$$\begin{aligned} [L_m, L_n] &= \frac{c}{12} m(m^2-1)\delta_{m,-n} + (m+1)\, L_{m+n} - (n+1) L_{m+n} \\ &= (m-n)\, L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n}. \end{aligned} \tag{42}$$

[1] See exercise 1 on assignment 4 for the first (painstakingly long) derivation.

> **Note:** We have not *really* rederived the Virasoro algebra here. Instead the operator product expansion (25) was really constructed in such a way as to reproduce this known commutator of two Virasoro generators.

c) The change of chiral fields (in a two-dimensional quantum field theory) under infinitesimal conformal transformations can be calculated using the conformal Ward-Takahashi identity. For transformations of the form $\tilde{z} = z + \epsilon(z)$ it reads

$$\delta_\epsilon \mathcal{O}(z) = \oint_{C_z} \frac{\mathrm{d}z'}{2\pi i}\, \epsilon(z')\, \mathcal{R}[T(z')\, \mathcal{O}(z)]. \tag{43}$$

Since we are interested in the transformational behavior of the energy-momentum tensor, we can again use the operator product expansion

$$\mathcal{R}[T(z')\, T(z)] = \frac{c/2}{(z'-z)^4} + \frac{2\,T(z)}{(z'-z)^2} + \frac{\partial_z T(z)}{z'-z} + \underbrace{\mathcal{O}[(z'-z)^0]}_{\text{finite terms}}, \tag{44}$$

as well as the Cauchy-Riemann formula (26) to compute

$$\begin{aligned} \delta_\epsilon T(z) &= \oint_{C_z} \frac{\mathrm{d}z'}{2\pi i}\, \epsilon(z')\, \mathcal{R}[T(z')\, T(z)] = \oint_{C_z} \frac{\mathrm{d}z'}{2\pi i} \epsilon(z') \left[\frac{\partial_z T(z)}{z'-z} + \frac{2\,T(z)}{(z'-z)^2} + \frac{c/2}{(z'-z)^4}\right] \\ &= \epsilon(z) \partial_z T(z) + 2\, [\partial_z \epsilon(z)]\, T(z) + \frac{c}{12}\, \partial_z^3 \epsilon(z), \end{aligned} \tag{45}$$

which is indeed the transformation we were asked to derive. Equation (45) confirms that $T(z)$ transforms as a tensor of weight $(2,0)$ but only for special transformations that satisfy $\partial_z^3 \epsilon(z) = 0$ or for general transformations if $c = 0$.

> **Note:** Instead of doing all the expansions by hand in section 1, part b), we could equally well have used the Ward-Takahashi identity there as well to calculate the change of primary fields under conformal transformations.

d) The Schwarzian derivative of the map $\omega \to z(\omega) = e^{\frac{2\pi}{l}\omega}$ from the cylinder to the plane reads

$$\begin{aligned} S(z,\omega) &= \frac{\partial^3 z}{\partial^3 \omega} \left(\frac{\partial z}{\partial \omega}\right)^{-1} - \frac{3}{2} \left(\frac{\partial^2 z}{\partial^2 \omega}\right)^2 \left(\frac{\partial z}{\partial \omega}\right)^{-2} \\ &= \left(\tfrac{2\pi}{l}\right)^3 z \left(\tfrac{2\pi}{l}\right)^{-1} z^{-1} - \tfrac{3}{2} \left(\tfrac{2\pi}{l}\right)^4 z^2 \left(\tfrac{2\pi}{l}\right)^{-2} z^{-2} = -\tfrac{1}{2} \left(\tfrac{2\pi}{l}\right)^2 \end{aligned} \tag{46}$$

Solving eq. (29) for the energy-momentum tensor on the cylinder yields

$$T_{\text{pln}}(z) = \left(\frac{\partial z}{\partial \omega}\right)^{-2} \left[T_{\text{cyl}}(\omega) - \frac{c}{12} S(z,\omega)\right] \qquad \Rightarrow \qquad T_{\text{cyl}}(\omega) = \left(\frac{\partial z}{\partial \omega}\right)^2 T_{\text{pln}}(z) + \frac{c}{12} S(z,\omega). \tag{47}$$

We plug in our result for the Schwarzian derivative and get

$$T_{\text{cyl}}(\omega) = \left(\tfrac{2\pi}{l}\right)^2 z^2\, T_{\text{pln}}(z) - \tfrac{1}{2} \left(\tfrac{2\pi}{l}\right)^2 \frac{c}{12} = \left(\tfrac{2\pi}{l}\right)^2 \left(z^2\, T_{\text{pln}}(z) - \frac{c}{24}\right). \tag{48}$$

We interpret the central term c of the quantum Virasoro algebra as the Casimir energy.

> **Note:** Operating on the plane means dealing with flat infinitely extended spacetime. Intuitively, the lowest possible energy, i.e. the zero-point energy $\langle H_0 \rangle$, on this space should vanish (see lecture notes, p. 81). Since Hamiltonian and energy-momentum tensor are directly related, $H \propto \int \mathrm{d}\sigma\, T(\tau,\sigma)$, this particularly implies that the one-point function on the plane must also

vanish, i.e. $\langle T_{\rm pln}(z)\rangle = 0$. Using this information in eq. (48), we get the very interesting result

$$\langle T_{\rm cyl}(\omega)\rangle = -\frac{c}{24}\left(\tfrac{2\pi}{l}\right)^2 . \tag{49}$$

Not only does eq. (49) demonstrate, that the one-point function on the cylinder is non-vanishing, it also seems to introduce a length scale into our theory, thereby breaking the conformal symmetry it enjoyed thus far. This is due to the underlying geometry containing a length scale: the radius l of the cylinder.

3 Bonus question: Fractional linear transformations

The purpose of this exercise is to show that conformal transformations on the Riemann sphere (i.e. on $\mathbb{C}\cup\{\infty\}$) map any 3 points to any other 3 points. The conformal group on the Riemann-sphere is given by $SL(2,\mathbb{C})$ of 2×2-matrices with unit determinant acting on the Riemann-sphere by so-called fractional linear transformations

$$z \to z' = \frac{az+b}{cz+d}, \tag{50}$$

where

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}). \tag{51}$$

a) Show that two successive fractional linear transformations,

$$z \to z' = \frac{az+b}{cz+d}, \qquad z' \to z'' = \frac{ez'+f}{gz'+h}, \tag{52}$$

are equivalent to one fractional linear transformation

$$z \to z'' = \frac{jz'+k}{lz'+m}, \tag{53}$$

where the matrix

$$\begin{pmatrix} j & k \\ l & m \end{pmatrix} \in SL(2,\mathbb{C}) \tag{54}$$

is the product of the two $SL(2,\mathbb{C})$ matrices that correspond to the single transformations $z \to z'$ and $z' \to z''$.

b) Show that the fractional linear action of the inverse matrix of (51) on z' leads back to z, and hence corresponds to the inverse transformation $z' \to z$.

c) Consider the map

$$z \to z' = \frac{(b-c)(z-a)}{(b-a)(z-c)}. \tag{55}$$

Show that this defines, up to an overall rescaling, an $SL(2,\mathbb{C})$ transformation provided a, b, c are pairwise distinct. Use this to show that $SL(2,\mathbb{C})$ maps any 3 distinct points on S^2 to any other 3 distinct points.

d) Show that the cross-ratio

$$\langle z_1, z_2, z_3, z_4\rangle \equiv \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} \tag{56}$$

is $SL(2,\mathbb{C})$ invariant. Use this to show that

$$\langle z, a, b, c\rangle = z', \tag{57}$$

where z' is the one given in part c).

a) The conformal group on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is the group of special linear transformations of degree 2,

$$SL(2,\mathbb{C}) = \{\boldsymbol{A} \in \mathrm{Mat}(2\times 2,\mathbb{C}) | \det(\boldsymbol{A}) = 1\}, \tag{58}$$

with ordinary matrix multiplication and matrix inversion as group operations. $SL(2,\mathbb{C})$ can be parametrized by fractional-linear transformations (also known as Möbius transformations)

$$z \to z' = \frac{az+b}{cz+d}, \qquad \boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}). \tag{59}$$

Performing two successive fractional linear transformations, $z \to z' = \frac{az+b}{cz+d}$ followed by $z' \to z'' = \frac{ez'+f}{gz'+h}$, we get

$$z'' = \frac{e\,\frac{az+b}{cz+d} + f}{g\,\frac{az+b}{cz+d} + h} = \frac{aez + be + cfz + df}{agz + bg + chz + dh} = \frac{(ae+cf)z + (be+df)}{(ag+ch)z + (bg+dh)} \equiv \frac{jz+k}{lz+m}. \tag{60}$$

Thus, the overall transformation is given by

$$\boldsymbol{C} = \begin{pmatrix} j & k \\ l & m \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}. \tag{61}$$

Since

$$\begin{aligned} \det(\boldsymbol{C}) &= (ae+cf)(bg+dh) - (ag+ch)(be+df) \\ &= \underline{abeg} + aedh + cfbg + \underline{\underline{cfdh}} - \underline{agbe} - agdf - chbe - \underline{\underline{chdf}} \\ &= aedh + cfbg - agdf - chbe = ad\underbrace{(eh-gf)}_{1} + bc\underbrace{(fg-eh)}_{-1} = ad - bc = 1, \qquad \checkmark \end{aligned} \tag{62}$$

$\boldsymbol{C}$ is also in $SL(2,\mathbb{C})$. Hence, any two successive $SL(2,\mathbb{C})$-transformations give a third. This is not surprising as we had already asserted $SL(2,\mathbb{C})$ to be a group.

b) The inverse of $\boldsymbol{A}$ from eq. (51) is given by

$$\boldsymbol{A}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{63}$$

Applying the fractional linear transformation corresponding to $\boldsymbol{A}^{-1}$ to z' returns z,

$$\frac{dz'-b}{-cz'+a} = \frac{d\frac{az+b}{cz+d} - b}{-c\frac{az+b}{cz+d} + a} = \frac{daz + \underline{db} - bcz - \underline{bd}}{\underline{\underline{-acz}} - cb + \underline{\underline{acz}} + ad} = \frac{(ad-bc)z}{ad-cb} = z. \qquad \checkmark \tag{64}$$

The inverse of transformation (50) is indeed represented by $\boldsymbol{A}^{-1}$.

c) Reshaped into the standard form of a fractional linear transformation, eq. (55) reads

$$z \to z' = \frac{(b-c)z - a(b-c)}{(b-a)z - c(b-a)}. \tag{65}$$

The corresponding $SL(2,\mathbb{C})$-matrix

$$\boldsymbol{D} = \begin{pmatrix} b-c & (c-b)a \\ b-a & (a-b)c \end{pmatrix} \tag{66}$$

has determinant

$$\det(\boldsymbol{D}) = (b-c)(a-b)c - (b-a)(c-b)a = (a-b)(b-c)(c-a). \tag{67}$$

As long as $a \neq b \wedge a \neq c \wedge b \neq c$, $\boldsymbol{D}$ has non-vanishing determinant and can be made $SL(2,\mathbb{C})$ with a simple scale factor of $\boldsymbol{D}' = \boldsymbol{D}/\sqrt{\det(\boldsymbol{D})}$.

As for showing that $SL(2,\mathbb{C})$ maps any 3 distinct points on S^2 to any other 3 distinct points, we note that the transformation (55) maps

$$z' = \frac{(b-c)(z-a)}{(b-a)(z-c)} = \begin{cases} 0 & \text{for } z = a, \\ 1 & \text{for } z = b, \\ \infty & \text{for } z = c, \end{cases} \tag{68}$$

where $\{0, 1, \infty\}$ all lie on S^2 and we already showed that a, b, and c are arbitrary but distinct points on S^2. Thus, all that remains to be shown is that there exists an $SL(2,\mathbb{C})$-transformation that maps $\{0, 1, \infty\}$ to any three distinct points $\{z_1, z_2, z_3\}$ on S^2. Fortunately, we already know that a transformation of the form (55), i.e.

$$z \to z' = \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}, \qquad \text{with} \begin{pmatrix} z_2 - z_3 & (z_3 - z_2)z_1 \\ z_2 - z_1 & (z_1 - z_2)z_3 \end{pmatrix} \equiv \boldsymbol{E} \tag{69}$$

maps $\{z_1, z_2, z_3\}$ to $\{0, 1, \infty\}$ and can be made $SL(2,\mathbb{C})$ by scaling

$$SL(2,\mathbb{C}) \quad \ni \quad \boldsymbol{E}' = \frac{1}{\sqrt{\det(\boldsymbol{E})}}\boldsymbol{E} = \frac{1}{\sqrt{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}}\boldsymbol{E}. \tag{70}$$

Thus the combined transformation $\boldsymbol{F} \equiv \boldsymbol{E}'^{-1}\boldsymbol{D}'$ maps any three points $\{a, b, c\}$ to any other three points $\{z_1, z_2, z_3\}$ on S^2, where the points in each set are pairwise distinct.

d) A simple calculation shows that the cross-ratio defined in eq. (56) is invariant under $SL(2,\mathbb{C})$-transformations,

$$\begin{aligned} \langle z_1, z_2, z_3, z_4\rangle \to \langle z_1', z_2', z_3', z_4'\rangle &= \frac{(z_1' - z_2')(z_3' - z_4')}{(z_1' - z_4')(z_3' - z_2')} \overset{(55)}{=} \frac{\left(\frac{z_1-a}{z_1-c} - \frac{z_2-a}{z_2-c}\right)\left(\frac{z_3-a}{z_3-c} - \frac{z_4-a}{z_4-c}\right)}{\left(\frac{z_1-a}{z_1-c} - \frac{z_4-a}{z_4-c}\right)\left(\frac{z_3-a}{z_3-c} - \frac{z_2-a}{z_2-c}\right)} \\ &= \frac{\frac{(z_1-z_2)(z_3-z_4)(a-c)^2}{(z_1-c)(z_2-c)(z_3-c)(z_4-c)}}{\frac{(z_1-z_4)(z_3-z_2)(a-c)^2}{(z_1-c)(z_2-c)(z_3-c)(z_4-c)}} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \langle z_1, z_2, z_3, z_4\rangle. \end{aligned} \tag{71}$$

Completely unrelated and simply by definition, we have

$$\langle z, a, b, c\rangle \overset{(56)}{=} \frac{(z-a)(b-c)}{(z-c)(b-a)} \overset{(55)}{=} z'. \tag{72}$$