

String Theory

Solution to Assignment 7

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1 Primary fields and radial quantization

A primary field $\phi(z, \bar{z})$ is a tensor field under conformal transformations $z \rightarrow z'$, $\bar{z} \rightarrow \bar{z}'$ in the sense that

$$\phi(z, \bar{z}) \rightarrow \phi'(z', \bar{z}') = \left(\frac{\partial z'}{\partial z} \right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \right)^{-\bar{h}} \phi(z, \bar{z}), \quad (1)$$

where $z, \bar{z}$ denote arbitrary holomorphic functions.

- a) How does a primary field with conformal weights $h, \bar{h}$ transform under dilations $z \rightarrow e^\lambda z$ and rotations $z \rightarrow e^{i\theta} z$ with $\lambda, \theta \in \mathbb{R}$? Give a physical interpretation for $h + \bar{h}$ and $h - \bar{h}$.
- b) Consider the infinitesimal conformal transformation

$$z'(z) = z + \zeta(z), \quad \bar{z}'(\bar{z}) = \bar{z} + \bar{\zeta}(\bar{z}). \quad (2)$$

Show that the infinitesimal transformation of a primary field is given by

$$\delta_{\zeta, \bar{\zeta}} \phi(z, \bar{z}) = \left[h \partial_z \zeta + \bar{h} \partial_{\bar{z}} \bar{\zeta} + \zeta \partial_z + \bar{\zeta} \partial_{\bar{z}} \right] \phi(z, \bar{z}). \quad (3)$$

- c) On the cylinder with coordinates (τ, σ) we defined

$$\xi^\pm = \tau \pm \sigma = -i(\tau' \pm i\sigma), \quad \tau = -i\tau', \quad (4)$$

and τ' is the time in Euclidean signature (after Wick rotation). After relabelling $\tau' \rightarrow \tau$ we have

$$\xi^+ = -i\bar{\omega}, \quad \xi^- = -i\omega, \quad \omega = \tau - i\sigma. \quad (5)$$

Argue that the conformal map from the cylinder to the Riemann sphere, defined by

$$\omega \rightarrow z(\omega) = e^{\frac{2\pi}{l}\omega}, \quad (6)$$

maps left-/right-moving fields to holomorphic/antiholomorphic fields and that for a primary field $\Phi = \Phi_L(\xi^-) + \Phi_R(\xi^+)$ a mode expansion

$$\Phi_L(\xi^-) = \left(\frac{2\pi}{l} \right)^h \sum_n \phi_n e^{i \frac{2\pi}{l} n \xi^-}, \quad \Phi_R(\xi^+) = \left(\frac{2\pi}{l} \right)^{\bar{h}} \sum_n \tilde{\phi}_n e^{i \frac{2\pi}{l} n \xi^+}, \quad (7)$$

translates into $\Phi(z, \bar{z}) = \Phi(z) + \bar{\Phi}(\bar{z})$,

$$\Phi(z) = \sum_n \phi_n z^{-n-h}, \quad \bar{\Phi}(\bar{z}) = \sum_n \phi_n \bar{z}^{-n-\bar{h}}. \quad (8)$$

d) Argue that a time-ordered product of fields on the cylinder maps to a radially ordered product

$$\mathcal{R}[\Phi_1(z_1)\Phi_2(z_2)] = \begin{cases} \Phi_1(z_1)\Phi_2(z_2) & \text{for } |z_1| > |z_2|, \\ \Phi_2(z_2)\Phi_1(z_1) & \text{for } |z_2| > |z_1|. \end{cases} \quad (9)$$

a) Under dilations (rescalings) $z \rightarrow z' = e^\lambda z$, $\lambda \in \mathbb{R}$, a primary field $\phi(z, \bar{z})$ with conformal weights $h, \bar{h}$ transforms as

$$\phi(z, \bar{z}) \rightarrow \phi'(z', \bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) = e^{-h\lambda} e^{-\bar{h}\lambda} \phi(z, \bar{z}) = e^{-(h+\bar{h})\lambda} \phi(z, \bar{z}). \quad (10)$$

Under rotations $z \rightarrow z' = e^{i\theta} z$, $\theta \in \mathbb{R}$, $\phi(z, \bar{z})$ becomes

$$\phi(z, \bar{z}) \rightarrow \phi'(z', \bar{z}') = e^{-ih\theta} e^{i\bar{h}\theta} \phi(z, \bar{z}) = e^{-i(h-\bar{h})\theta} \phi(z, \bar{z}). \quad (11)$$

From eqs. (10) and (11), we infer that the sum of conformal weights $h + \bar{h} = \Delta$ gives the **scaling dimension** of a primary field $\phi(z, \bar{z})$, whereas the difference $h - \bar{h} = s$ is the field's (conformal) **spin**.

b) Equation (1) can equivalently be written as

$$\phi(z, \bar{z}) \rightarrow \phi'(z, \bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z', \bar{z}'). \quad (12)$$

For the conformal transformations $z'(z) = z + \zeta(z)$, $\bar{z}'(\bar{z}) = \bar{z} + \bar{\zeta}(\bar{z})$, we can expand each term in eq. (12) to first order in the infinitesimal (anti-)chiral fields $\zeta(z)$, $\bar{\zeta}(\bar{z})$ to get

$$\left(\frac{\partial z'}{\partial z}\right)^{-h} = \left(1 + \frac{\partial \zeta(z)}{\partial z}\right)^{-h} = 1 + h \partial_z \zeta(z) + \mathcal{O}(\zeta^2), \quad (13)$$

$$\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} = \left(1 + \frac{\partial \bar{\zeta}(\bar{z})}{\partial \bar{z}}\right)^{-\bar{h}} = 1 + \bar{h} \partial_{\bar{z}} \bar{\zeta}(\bar{z}) + \mathcal{O}(\bar{\zeta}^2), \quad (14)$$

$$\phi(z', \bar{z}') = \phi(z, \bar{z}) + \zeta(z) \partial_z \phi(z, \bar{z}) + \bar{\zeta}(\bar{z}) \partial_{\bar{z}} \phi(z, \bar{z}) + \mathcal{O}(\zeta^2, \bar{\zeta}^2). \quad (15)$$

Dropping all terms beyond linear order gives the infinitesimal transformation

$$\begin{aligned} \delta_{\zeta, \bar{\zeta}} \phi(z, \bar{z}) &\equiv \phi'(z, \bar{z}) - \phi(z, \bar{z}) \\ &= \left(1 + h \partial_z \zeta(z)\right) \left(1 + \bar{h} \partial_{\bar{z}} \bar{\zeta}(\bar{z})\right) \left(1 + \zeta(z) \partial_z + \bar{\zeta}(\bar{z}) \partial_{\bar{z}}\right) \phi(z, \bar{z}) - \phi(z, \bar{z}) \\ &= \left[1 + h \partial_z \zeta(z) + \bar{h} \partial_{\bar{z}} \bar{\zeta}(\bar{z}) + \zeta(z) \partial_z + \bar{\zeta}(\bar{z}) \partial_{\bar{z}}\right] \phi(z, \bar{z}) - \phi(z, \bar{z}) \\ &= \left[h \partial_z \zeta(z) + \bar{h} \partial_{\bar{z}} \bar{\zeta}(\bar{z}) + \zeta(z) \partial_z + \bar{\zeta}(\bar{z}) \partial_{\bar{z}}\right] \phi(z, \bar{z}). \end{aligned} \quad (16)$$

c) A Wick rotation on the cylinder with coordinates $(\tau, \sigma) \in \mathbb{R}^2$ sends

$$\xi^\pm = \tau \pm \sigma \xrightarrow{\text{W.R.}} -i(\tau \pm i\sigma) \equiv \begin{cases} -i\bar{\omega} = \xi^+, \\ -i\omega = \xi^-, \end{cases} \quad (17)$$

where now $\tau \in i\mathbb{R}$ and $\omega, \bar{\omega}$ are coordinates on the cylinder.

After Wick rotating, the conformal map

$$i\xi^- = \omega \rightarrow z(\omega) = e^{\frac{2\pi}{l}\omega}, \quad i\xi^+ = \bar{\omega} \rightarrow \bar{z}(\bar{\omega}) = e^{\frac{2\pi}{l}\bar{\omega}}. \quad (18)$$

from the cylinder to the Riemann sphere respectively the complex plane transforms left-movers into holomorphic fields $\Phi_L(\xi^-) \rightarrow \Phi(z(\omega))$ ones and right-movers into antiholomorphic ones $\Phi_R(\xi^+) \rightarrow \Phi(\bar{z}(\bar{\omega}))$.

This can be illustrated for the case of a primary field $\Phi(\xi^+, \xi^-) = \Phi_L(\xi^-) + \Phi_R(\xi^+)$ with conformal weights $(h, \bar{h})$, where the left- and right-moving components are given by

$$\Phi_L(\xi^-) = \left(\frac{2\pi}{l}\right)^h \sum_n \phi_n e^{i\frac{2\pi}{l}n\xi^-}, \quad \Phi_R(\xi^+) = \left(\frac{2\pi}{l}\right)^{\bar{h}} \sum_n \tilde{\phi}_n e^{i\frac{2\pi}{l}n\xi^+}. \quad (19)$$

Under (18), these transform into

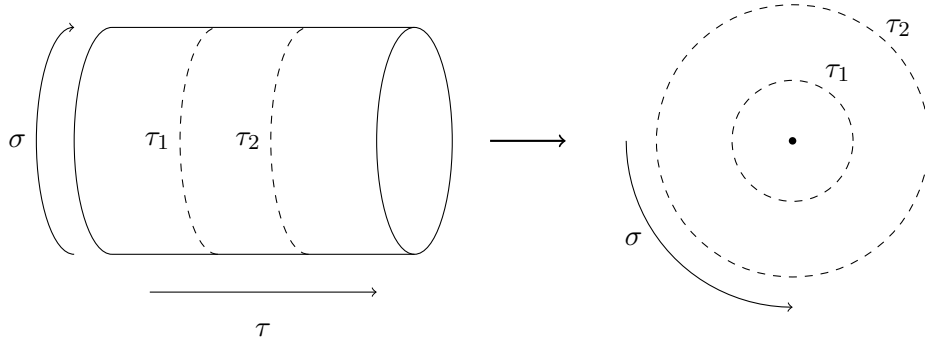
$$\begin{aligned} \Phi_L(\xi^-) \rightarrow \Phi_L(z) &= \left(\frac{\partial z}{\partial \omega}\right)^{-h} \Phi_L(\xi^-) = \left(\frac{2\pi}{l} e^{\frac{2\pi}{l}\omega}\right)^{-h} \left(\frac{2\pi}{l}\right)^h \sum_n \phi_n e^{-\frac{2\pi}{l}n\omega} \\ &= \sum_n \phi_n e^{(\frac{2\pi}{l}\omega)^{-n-h}} = \sum_n \phi_n z^{-n-h}, \end{aligned} \quad (20)$$

$$\begin{aligned} \Phi_R(\xi^+) \rightarrow \Phi_R(\bar{z}) &= \left(\frac{\partial \bar{z}}{\partial \bar{\omega}}\right)^{-\bar{h}} \Phi_R(\xi^+) = \left(\frac{2\pi}{l} e^{\frac{2\pi}{l}\bar{\omega}}\right)^{-\bar{h}} \left(\frac{2\pi}{l}\right)^{\bar{h}} \sum_n \tilde{\phi}_n e^{-\frac{2\pi}{l}n\bar{\omega}} \\ &= \sum_n \tilde{\phi}_n e^{(\frac{2\pi}{l}\bar{\omega})^{-n-\bar{h}}} = \sum_n \tilde{\phi}_n \bar{z}^{-n-\bar{h}}. \end{aligned} \quad (21)$$

- d) To see that time ordering on the cylinder corresponds to radial ordering on the complex plane, we choose any two points $\omega_1 = \tau_1 - i\sigma_1$, $\omega_2 = \tau_2 - i\sigma_2$ such that $\tau_1 \leq \tau_2$, i.e. in a time-ordered operator product containing τ_1 and τ_2 , the operator evaluated at τ_1 would act first. For the corresponding points z_1, z_2 on the plane, we have the relation

$$\begin{aligned} \tau_1 \leq \tau_2 \quad \Longleftrightarrow \quad |z_1| &= \left| e^{\frac{2\pi}{l}\omega_1} \right| = \left| e^{\frac{2\pi}{l}(\tau_1 - i\sigma_1)} \right| = e^{\frac{2\pi}{l}\tau_1} \\ &\leq e^{\frac{2\pi}{l}\tau_2} = \left| e^{\frac{2\pi}{l}(\tau_2 - i\sigma_2)} \right| = |z_2|. \end{aligned} \quad (22)$$

Thus in a radially ordered product, an operator evaluated on the plane at z_1 would also act first. The following graphic visualizes how different times on the cylinder correspond to different times on the plane.



2 The OPE of the energy momentum tensor

In the complex plane, the Virasoro generators L_n , $n \in \mathbb{Z}$ are given by

$$L_n = \oint_{C_0} \frac{dz}{2\pi i} z^{n+1} T(z). \quad (23)$$

a) Show that

$$[L_m, L_n] = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} w^{n+1} \mathcal{R}[T(z) T(w)], \quad (24)$$

where C_0 denotes a contour about $z = 0$ and C_w is a contour about $z = w$. As usual the product $\mathcal{R}[T(z) T(w)]$ is meant to be the radially ordered product.

Hint: Write the commutator as a difference of two double contour integrals and use a contour deformation of the dz integration for fixed w .

b) Use eq. (24) and the (radially ordered) operator product

$$\mathcal{R}[T(z) T(w)] = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \underbrace{\mathcal{O}[(z-w)^0]}_{\text{finite terms}}, \quad (25)$$

as well as the Cauchy-Riemann formula,

$$\oint_{C_w} \frac{dz'}{2\pi i} \frac{f(z')}{(z'-z)^n} = \frac{1}{(n-1)!} f^{(n-1)}(z), \quad (26)$$

to rederive the quantum Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}. \quad (27)$$

c) The Schwarzian derivative is defined as

$$S(\tilde{z}, z) = \frac{\partial^3 \tilde{z}}{\partial^3 z} \left(\frac{\partial \tilde{z}}{\partial z} \right)^{-1} - \frac{3}{2} \left(\frac{\partial^2 \tilde{z}}{\partial^2 z} \right)^2 \left(\frac{\partial \tilde{z}}{\partial z} \right)^{-2}. \quad (28)$$

Show that the transformation

$$T(z) \rightarrow \tilde{T}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z} \right)^{-2} \left[T(z) - \frac{c}{12} S(\tilde{z}, z) \right], \quad (29)$$

gives at the infinitesimal level for $\tilde{z} = z + \epsilon(z)$,

$$\delta T(z) = \epsilon(z) \partial T(z) + 2[\partial_z \epsilon(z)] T(z) + \frac{c}{12} \partial_z^3 \epsilon(z). \quad (30)$$

d) Apply this to the map from the cylinder to the plane, given by

$$\tau - i\sigma \equiv \omega \rightarrow z(\omega) = e^{\frac{2\pi}{l}\omega}, \quad (31)$$

to show that

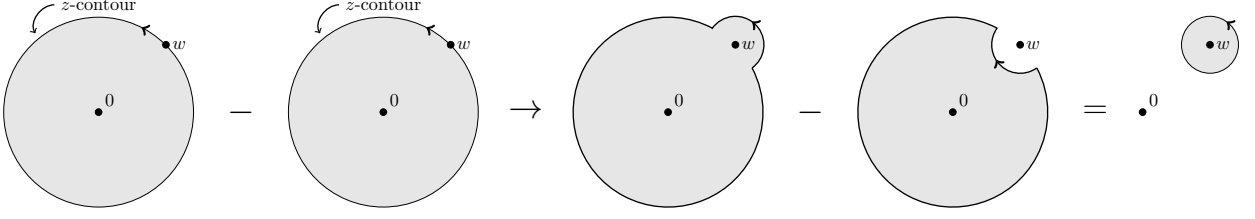
$$T_{\text{cyl}}(\omega) = \left(\frac{2\pi}{l} \right)^2 \left(z^2 T_{\text{pln}}(z) - \frac{c}{24} \right). \quad (32)$$

Which are the various physical interpretations for c ?

a) Expanding the commutator of Virasoro generators gives

$$\begin{aligned} [L_m, L_n] &= \oint_{C_0} \frac{dz}{2\pi i} z^{m+1} T(z) \oint_{C_0} \frac{dw}{2\pi i} w^{n+1} T(w) - \oint_{C_0} \frac{dw}{2\pi i} w^{n+1} T(w) \oint_{C_0} \frac{dz}{2\pi i} z^{m+1} T(z) \\ &= \oint_{C_0} \frac{dw}{2\pi i} \left(\oint_{C_0} \frac{dz}{2\pi i} z^{m+1} w^{n+1} T(z) T(w) - \oint_{C_0} \frac{dz}{2\pi i} z^{m+1} w^{n+1} T(w) T(z) \right). \end{aligned} \quad (33)$$

We use the following contour deformation for the two integrals in parenthesis so that *after* the deformation, we have $|w| \leq |z| \ \forall z$ in the first integral and $|w| \geq |z| \ \forall z$ in the second.



Then eq. (33) simplifies to

$$[L_m, L_n] = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} w^{n+1} \mathcal{R}[T(z) T(w)]. \quad (34)$$

b) To rederive¹ the quantum Virasoro algebra, we insert eq. (25) into eq. (34) to get

$$[L_m, L_n] = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} w^{n+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \right] \quad (35)$$

We can perform the z -integration using the Cauchy-Riemann formula

$$\oint_{C_w} \frac{dz'}{2\pi i} \frac{f(z')}{(z'-z)^n} = \frac{1}{(n-1)!} f^{(n-1)}(z). \quad (36)$$

The three terms in brackets thus become

$$\oint_{C_w} \frac{dz}{2\pi i} \frac{\frac{c}{2} z^{m+1} w^{n+1}}{(z-w)^4} = \frac{c}{2 \cdot 3!} (m+1)m(m-1)w^{m-2}w^{n+1} = \frac{c}{12} m(m^2-1)w^{m-n-1}, \quad (37)$$

$$\oint_{C_w} \frac{dz}{2\pi i} \frac{2T(w) z^{m+1} w^{n+1}}{(z-w)^2} = 2T(w) (m+1)w^m w^{n+1}, \quad (38)$$

$$\oint_{C_w} \frac{dz}{2\pi i} \frac{\partial_w T(w) z^{m+1} w^{n+1}}{z-w} = \partial_w T(w) w^{m+1} w^{n+1}. \quad (39)$$

We reinsert eqs. (37) to (39) into eq. (35) and get

$$\begin{aligned} [L_m, L_n] &= \oint_{C_0} \frac{dw}{2\pi i} \left[\frac{c}{12} m(m^2-1)w^{m-n-1} + 2T(w) (m+1)w^m w^{n+1} + \partial_w T(w) w^{m+1} w^{n+1} \right] \\ &= \frac{c}{12} m(m^2-1)\delta_{m,-n} + \underbrace{\oint_{C_0} \frac{dw}{2\pi i} \left[T(w) (m+1)w^m w^{n+1} + \partial_w [T(w) w^{m+1}] w^{n+1} \right]}_{(m+1)L_{m+n}} \end{aligned} \quad (40)$$

Integration by parts in the last term where the boundary terms cancel since we integrate over the closed contour C_0 gives

$$\begin{aligned} \oint_{C_0} \frac{dw}{2\pi i} \partial_w [T(w) w^{m+1}] w^{n+1} &= - \oint_{C_0} \frac{dw}{2\pi i} T(w) w^{m+1} \partial_w w^{n+1} \\ &= -(n+1) \oint_{C_0} \frac{dw}{2\pi i} T(w) w^{m+1} w^n = -(n+1)L_{m+n}. \end{aligned} \quad (41)$$

Thus we arrive at the renowned Virasoro algebra

$$\begin{aligned} [L_m, L_n] &= \frac{c}{12} m(m^2-1)\delta_{m,-n} + (m+1)L_{m+n} - (n+1)L_{m+n} \\ &= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n}. \end{aligned} \quad (42)$$

¹See [exercise 1 on assignment 4](#) for the first (painstakingly long) derivation.

Note: We have not *really* rederived the Virasoro algebra here. Instead the operator product expansion (25) was really constructed in such a way as to reproduce this known commutator of two Virasoro generators.

- c) The change of chiral fields (in a two-dimensional quantum field theory) under infinitesimal conformal transformations can be calculated using the conformal Ward-Takahashi identity. For transformations of the form $\tilde{z} = z + \epsilon(z)$ it reads

$$\delta_\epsilon \mathcal{O}(z) = \oint_{C_z} \frac{dz'}{2\pi i} \epsilon(z') \mathcal{R}[T(z') \mathcal{O}(z)]. \quad (43)$$

Since we are interested in the transformational behavior of the energy-momentum tensor, we can again use the operator product expansion

$$\mathcal{R}[T(z') T(z)] = \frac{c/2}{(z' - z)^4} + \frac{2T(z)}{(z' - z)^2} + \frac{\partial_z T(z)}{z' - z} + \underbrace{\mathcal{O}[(z' - z)^0]}_{\text{finite terms}}, \quad (44)$$

as well as the Cauchy-Riemann formula (26) to compute

$$\begin{aligned} \delta_\epsilon T(z) &= \oint_{C_z} \frac{dz'}{2\pi i} \epsilon(z') \mathcal{R}[T(z') T(z)] = \oint_{C_z} \frac{dz'}{2\pi i} \epsilon(z') \left[\frac{\partial_z T(z)}{z' - z} + \frac{2T(z)}{(z' - z)^2} + \frac{c/2}{(z' - z)^4} \right] \\ &= \epsilon(z) \partial_z T(z) + 2[\partial_z \epsilon(z)] T(z) + \frac{c}{12} \partial_z^3 \epsilon(z), \end{aligned} \quad (45)$$

which is indeed the transformation we were asked to derive. Equation (45) confirms that $T(z)$ transforms as a tensor of weight $(2, 0)$ but only for special transformations that satisfy $\partial_z^3 \epsilon(z) = 0$ or for general transformations if $c = 0$.

Note: Instead of doing all the expansions by hand in section 1, part b), we could equally well have used the Ward-Takahashi identity there as well to calculate the change of primary fields under conformal transformations.

- d) The Schwarzian derivative of the map $\omega \rightarrow z(\omega) = e^{\frac{2\pi}{l}\omega}$ from the cylinder to the plane reads

$$\begin{aligned} S(z, \omega) &= \frac{\partial^3 z}{\partial^3 \omega} \left(\frac{\partial z}{\partial \omega} \right)^{-1} - \frac{3}{2} \left(\frac{\partial^2 z}{\partial^2 \omega} \right)^2 \left(\frac{\partial z}{\partial \omega} \right)^{-2} \\ &= \left(\frac{2\pi}{l} \right)^3 z \left(\frac{2\pi}{l} \right)^{-1} z^{-1} - \frac{3}{2} \left(\frac{2\pi}{l} \right)^4 z^2 \left(\frac{2\pi}{l} \right)^{-2} z^{-2} = -\frac{1}{2} \left(\frac{2\pi}{l} \right)^2 \end{aligned} \quad (46)$$

Solving eq. (29) for the energy-momentum tensor on the cylinder yields

$$T_{\text{pln}}(z) = \left(\frac{\partial z}{\partial \omega} \right)^{-2} \left[T_{\text{cyl}}(\omega) - \frac{c}{12} S(z, \omega) \right] \quad \Rightarrow \quad T_{\text{cyl}}(\omega) = \left(\frac{\partial z}{\partial \omega} \right)^2 T_{\text{pln}}(z) + \frac{c}{12} S(z, \omega). \quad (47)$$

We plug in our result for the Schwarzian derivative and get

$$T_{\text{cyl}}(\omega) = \left(\frac{2\pi}{l} \right)^2 z^2 T_{\text{pln}}(z) - \frac{1}{2} \left(\frac{2\pi}{l} \right)^2 \frac{c}{12} = \left(\frac{2\pi}{l} \right)^2 \left(z^2 T_{\text{pln}}(z) - \frac{c}{24} \right). \quad (48)$$

We interpret the central term c of the quantum Virasoro algebra as the Casimir energy.

Note: Operating on the plane means dealing with flat infinitely extended spacetime. Intuitively, the lowest possible energy, i.e. the zero-point energy $\langle H_0 \rangle$, on this space should vanish (see lecture notes, p. 81). Since Hamiltonian and energy-momentum tensor are directly related, $H \propto \int d\sigma T(\tau, \sigma)$, this particularly implies that the one-point function on the plane must also

vanish, i.e. $\langle T_{\text{pln}}(z) \rangle = 0$. Using this information in eq. (48), we get the very interesting result

$$\langle T_{\text{cyl}}(\omega) \rangle = -\frac{c}{24} \left(\frac{2\pi}{l} \right)^2. \quad (49)$$

Not only does eq. (49) demonstrate, that the one-point function on the cylinder is non-vanishing, it also seems to introduce a length scale into our theory, thereby breaking the conformal symmetry it enjoyed thus far. This is due to the underlying geometry containing a length scale: the radius l of the cylinder.

3 Bonus question: Fractional linear transformations

The purpose of this exercise is to show that conformal transformations on the Riemann sphere (i.e. on $\mathbb{C} \cup \{\infty\}$) map any 3 points to any other 3 points. The conformal group on the Riemann-sphere is given by $SL(2, \mathbb{C})$ of 2×2 -matrices with unit determinant acting on the Riemann-sphere by so-called fractional linear transformations

$$z \rightarrow z' = \frac{az + b}{cz + d}, \quad (50)$$

where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}). \quad (51)$$

a) Show that two successive fractional linear transformations,

$$z \rightarrow z' = \frac{az + b}{cz + d}, \quad z' \rightarrow z'' = \frac{ez' + f}{gz' + h}, \quad (52)$$

are equivalent to one fractional linear transformation

$$z \rightarrow z'' = \frac{jz' + k}{lz' + m}, \quad (53)$$

where the matrix

$$\begin{pmatrix} j & k \\ l & m \end{pmatrix} \in SL(2, \mathbb{C}) \quad (54)$$

is the product of the two $SL(2, \mathbb{C})$ matrices that correspond to the single transformations $z \rightarrow z'$ and $z' \rightarrow z''$.

b) Show that the fractional linear action of the inverse matrix of (51) on z' leads back to z , and hence corresponds to the inverse transformation $z' \rightarrow z$.

c) Consider the map

$$z \rightarrow z' = \frac{(b-c)(z-a)}{(b-a)(z-c)}. \quad (55)$$

Show that this defines, up to an overall rescaling, an $SL(2, \mathbb{C})$ transformation provided a, b, c are pairwise distinct. Use this to show that $SL(2, \mathbb{C})$ maps any 3 distinct points on S^2 to any other 3 distinct points.

d) Show that the cross-ratio

$$\langle z_1, z_2, z_3, z_4 \rangle \equiv \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \quad (56)$$

is $SL(2, \mathbb{C})$ invariant. Use this to show that

$$\langle z, a, b, c \rangle = z', \quad (57)$$

where z' is the one given in part c).

- a) The conformal group on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is the group of special linear transformations of degree 2,

$$SL(2, \mathbb{C}) = \{\mathbf{A} \in \text{Mat}(2 \times 2, \mathbb{C}) \mid \det(\mathbf{A}) = 1\}, \quad (58)$$

with ordinary matrix multiplication and matrix inversion as group operations. $SL(2, \mathbb{C})$ can be parametrized by fractional-linear transformations (also known as Möbius transformations)

$$z \rightarrow z' = \frac{az + b}{cz + d}, \quad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}). \quad (59)$$

Performing two successive fractional linear transformations, $z \rightarrow z' = \frac{az+b}{cz+d}$ followed by $z' \rightarrow z'' = \frac{ez'+f}{gz'+h}$, we get

$$z'' = \frac{e \frac{az+b}{cz+d} + f}{g \frac{az+b}{cz+d} + h} = \frac{aez + be + cfz + df}{agz + bg + chz + dh} = \frac{(ae + cf)z + (be + df)}{(ag + ch)z + (bg + dh)} \equiv \frac{jz + k}{lz + m}. \quad (60)$$

Thus, the overall transformation is given by

$$\mathbf{C} = \begin{pmatrix} j & k \\ l & m \end{pmatrix} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix}. \quad (61)$$

Since

$$\begin{aligned} \det(\mathbf{C}) &= (ae + cf)(bg + dh) - (ag + ch)(be + df) \\ &= \underline{abeg} + aedh + cfbg + \underline{cf dh} - \underline{agbe} - agdf - chbe - \underline{chdf} \\ &= aedh + cfbg - agdf - chbe = ad(\underbrace{eh - gf}_1) + bc(\underbrace{fg - eh}_{-1}) = ad - bc = 1, \quad \checkmark \end{aligned} \quad (62)$$

$\mathbf{C}$ is also in $SL(2, \mathbb{C})$. Hence, any two successive $SL(2, \mathbb{C})$ -transformations give a third. This is not surprising as we had already asserted $SL(2, \mathbb{C})$ to be a group.

- b) The inverse of $\mathbf{A}$ from eq. (51) is given by

$$\mathbf{A}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \quad (63)$$

Applying the fractional linear transformation corresponding to $\mathbf{A}^{-1}$ to z' returns z ,

$$\frac{dz' - b}{-cz' + a} = \frac{d \frac{az+b}{cz+d} - b}{-c \frac{az+b}{cz+d} + a} = \frac{daz + \underline{db} - bcz - \underline{bd}}{\underline{-acz} - cb + \underline{acz} + ad} = \frac{(ad - bc)z}{ad - cb} = z. \quad \checkmark \quad (64)$$

The inverse of transformation (50) is indeed represented by $\mathbf{A}^{-1}$.

- c) Reshaped into the standard form of a fractional linear transformation, eq. (55) reads

$$z \rightarrow z' = \frac{(b - c)z - a(b - c)}{(b - a)z - c(b - a)}. \quad (65)$$

The corresponding $SL(2, \mathbb{C})$ -matrix

$$\mathbf{D} = \begin{pmatrix} b - c & (c - b)a \\ b - a & (a - b)c \end{pmatrix} \quad (66)$$

has determinant

$$\det(\mathbf{D}) = (b - c)(a - b)c - (b - a)(c - b)a = (a - b)(b - c)(c - a). \quad (67)$$

As long as $a \neq b \wedge a \neq c \wedge b \neq c$, $\mathbf{D}$ has non-vanishing determinant and can be made $SL(2, \mathbb{C})$ with a simple scale factor of $\mathbf{D}' = \mathbf{D}/\sqrt{\det(\mathbf{D})}$.

As for showing that $SL(2, \mathbb{C})$ maps any 3 distinct points on S^2 to any other 3 distinct points, we note that the transformation (55) maps

$$z' = \frac{(b-c)(z-a)}{(b-a)(z-c)} = \begin{cases} 0 & \text{for } z = a, \\ 1 & \text{for } z = b, \\ \infty & \text{for } z = c, \end{cases} \quad (68)$$

where $\{0, 1, \infty\}$ all lie on S^2 and we already showed that a, b , and c are arbitrary but distinct points on S^2 . Thus, all that remains to be shown is that there exists an $SL(2, \mathbb{C})$ -transformation that maps $\{0, 1, \infty\}$ to any three distinct points $\{z_1, z_2, z_3\}$ on S^2 . Fortunately, we already know that a transformation of the form (55), i.e.

$$z \rightarrow z' = \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}, \quad \text{with } \begin{pmatrix} z_2 - z_3 & (z_3 - z_2)z_1 \\ z_2 - z_1 & (z_1 - z_2)z_3 \end{pmatrix} \equiv \mathbf{E} \quad (69)$$

maps $\{z_1, z_2, z_3\}$ to $\{0, 1, \infty\}$ and can be made $SL(2, \mathbb{C})$ by scaling

$$SL(2, \mathbb{C}) \ni \mathbf{E}' = \frac{1}{\sqrt{\det(\mathbf{E})}} \mathbf{E} = \frac{1}{\sqrt{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}} \mathbf{E}. \quad (70)$$

Thus the combined transformation $\mathbf{F} \equiv \mathbf{E}'^{-1} \mathbf{D}'$ maps any three points $\{a, b, c\}$ to any other three points $\{z_1, z_2, z_3\}$ on S^2 , where the points in each set are pairwise distinct.

- d) A simple calculation shows that the cross-ratio defined in eq. (56) is invariant under $SL(2, \mathbb{C})$ -transformations,

$$\begin{aligned} \langle z_1, z_2, z_3, z_4 \rangle &\rightarrow \langle z'_1, z'_2, z'_3, z'_4 \rangle = \frac{(z'_1 - z'_2)(z'_3 - z'_4)}{(z'_1 - z'_4)(z'_3 - z'_2)} \stackrel{(55)}{=} \frac{\left(\frac{z_1-a}{z_1-c} - \frac{z_2-a}{z_2-c}\right) \left(\frac{z_3-a}{z_3-c} - \frac{z_4-a}{z_4-c}\right)}{\left(\frac{z_1-a}{z_1-c} - \frac{z_4-a}{z_4-c}\right) \left(\frac{z_3-a}{z_3-c} - \frac{z_2-a}{z_2-c}\right)} \\ &= \frac{\frac{(z_1-z_2)(z_3-z_4)(a-c)^2}{(z_1-c)(z_2-c)(z_3-c)(z_4-c)}}{\frac{(z_1-z_4)(z_3-z_2)(a-c)^2}{(z_1-c)(z_2-c)(z_3-c)(z_4-c)}} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \langle z_1, z_2, z_3, z_4 \rangle. \end{aligned} \quad (71)$$

Completely unrelated and simply by definition, we have

$$\langle z, a, b, c \rangle \stackrel{(56)}{=} \frac{(z-a)(b-c)}{(z-c)(b-a)} \stackrel{(55)}{=} z'. \quad (72)$$