19.07.2015

General Relativity - Exercise Sheet 7

Problem 1 (Equation-of-state parameter meets scalar field) [15 points]

The continuity equation for the energy density of the cosmos reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w) = 0, \qquad (1)$$

with the eq.-of-state parameter $w = \frac{p}{c^2\rho}$. For w constant, it leads to an eq. describing the time evolution of the density of the form

$$\rho = \rho_0 a^{-3(1+w)}.$$

For Λ, we had $w_\Lambda = -1$, which means ρ_Λ constant. It has been suggested that $w_\Lambda = w_\Lambda(a)$.

a) Show that

$$\rho = \rho_0 \exp\left(-3\int_1^a d(\ln a')[1+w(a')]\right)$$

solves eq. (1) for a scale factor-dependent $w(a)$.

$$\dot{\rho} = \rho_0 \exp\left(-3\int_1^a d(\ln a')[1+w(a')]\right)\frac{d}{dt}\left(-3\int_1^a d(\ln a')[1+w(a')]\right),$$

where $d(\ln a') = \frac{da'}{a'} = \frac{1}{a'}\frac{da'}{dt}dt = \frac{\dot{a}'}{a'}dt$. Thus

$$\dot{\rho} = \rho\left(-3\int_1^a dt\,\frac{d}{dt}\,\frac{\dot{a}'}{a'}[1+w(a')]\right) = -3\rho\left(\frac{\dot{a}}{a}[1+w(a)] - [1+\underbrace{w(1)}_{-1}]\right)$$

$$= -3\frac{\dot{a}}{a}[1+w(a)],$$

where $w(1) = w(a(t_0)) = -1$ as stated above. Therefore,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w(a)) = \left[-3\frac{\dot{a}}{a}(1+w(a)) + 3\frac{\dot{a}}{a}(1+w(a))\right]\rho = 0. \quad \checkmark$$

b) We have calculated that for a scalar field ϕ with a Lagrangian $L = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$, the pressure and energy density can, under the assumption of homogeneity and isotropy of the field, be written

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \qquad \rho c^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi). \qquad (2)$$

Therefore, the eq.-of-state parameter w becomes

$$w = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)},$$

which, for a vanishing kinetic term $\dot{\phi} = 0$, mimics that of a cosmological constant, $w = -1$.

By plugging the densities and pressures of a scalar field into eq. (1), can you arrive at a e.o.m. for ϕ?

$$\dot{\rho} c^2 = \dot{\phi}\ddot{\phi} + \frac{\partial V(\phi)}{\partial \phi}\dot{\phi} = (\ddot{\phi} + V'(\phi))\,\dot{\phi}$$

Insertion into eq. (1) multiplied by c^2 yields

$$\dot{\rho} c^2 + 3\frac{\dot{a}}{a}\rho c^2 (1+w) = (\ddot{\phi} + V'(\phi))\,\dot{\phi} + 3\frac{\dot{a}}{a}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\left(1 + \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\right)$$

$$= (\ddot{\phi} + V'(\phi))\,\dot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi}^2 = 0$$

$$\Longrightarrow \quad 0 = \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi)$$

Since our investigation is concerned with a cosmos we assume to be isotropic and, in particular, homogeneous, we have $\phi = \phi(\cancel{x}, t)$ and thus $\vec{\nabla}\phi(t) = 0$.

Except for the term $3\frac{\dot{a}}{a}\dot{\phi}$, our result therefore markedly resembles the Klein-Gordon eq.

$$\Box\phi + \frac{\partial V(\phi)}{\partial \phi} = 0$$

of a scalar field in some potential $V(\phi)$.

19.07.2015

Problem 2 (Field equations) [15 points]

Show that the Einstein field equations,

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}, \quad (3)$$

could equally well be written as

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\right] + \Lambda g_{\mu\nu}. \quad (4)$$

By contracting all indices, eq. (3) becomes (we use $g^{\mu}{}_{\mu} = g^{\mu\nu} g_{\mu\nu} = \delta^{\mu}{}_{\mu} = 4$)

$$R - \frac{R}{2} 4 + 4\Lambda = -R + 4\Lambda = -\frac{8\pi G}{c^4} T,$$

where $R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$ and $T = T^{\mu}{}_{\mu} = g^{\mu\nu} T_{\mu\nu}$. We multiply our intermediate result with $-\frac{1}{2} g_{\mu\nu}$ and add it to the field eqs. (3) to get

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{R}{2} g_{\mu\nu} - 2\Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\right),$$

which is equivalent to

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\right) + \Lambda g_{\mu\nu}.$$

Problem 3 (A Hubble diagram for the 21st century) [10 points]

a) We cannot distinguish between cosmic and Doppler-redshift. How do we know that most receding objects follow the cosmic flow and are not actively moving away?

Good question.

Problem 4 (Extra: CPT symmetry) [5 points]

Are the Einstein field CPT-invariant? I.e. do physical processes stay the same if one were to transform $x^\mu \to -x^\mu$ and $q \to -q$?

Yes, both sides of

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\right) + g_{\mu\nu}\Lambda$$

are CPT-even. This implies that an antimatter energy-momentum tensor generates a gravitational field, i.e. spacetime curvature, in the same way as one for matter does.