29.06.2015

Quantum Field Theory II – Assignment 10

Problem 10.1 (One-loop β-function in QED)

a) Compute the one-loop β-function in QED.

$$\beta_e(\mu) = \mu \frac{\partial}{\partial \mu}\left(-e\delta_1 + e\delta_2 + \frac{e}{2}\delta_3\right), \quad \text{where } -\mu^2 = 4m^2, \text{ i.e. } \mu \propto m$$

$$\delta_1 = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz\,(1-z)\left(\frac{\Gamma(2-\frac{d}{2})}{((1-z)^2 m^2 + z M^2)^{2-\frac{d}{2}}}\frac{1}{2}(2-\epsilon)^2 + \frac{\Gamma(3-\frac{d}{2})}{((1-z)^2 m^2 + z M^2)^{3-\frac{d}{2}}}\left(2(1-4z+z^2)-\epsilon(1-z)^2\right)m^2\right)$$

$$\frac{\partial \delta_1}{\partial \mu} = -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dz\,(1-z)\left[\frac{\overbrace{-(2-\frac{d}{2})\Gamma(2-\frac{d}{2})}^{\Gamma(3-\frac{d}{2}) = \Gamma(1+\frac{\epsilon}{2}) = 1}}{((1-z)^2 m^2 + z M^2)^{3-\frac{d}{2}}} 2(1-z)^2 m \frac{1}{2}(2-\epsilon)^2 + \frac{\partial}{\partial \mu}\frac{\Gamma(1)}{(1-z)^2 + z\frac{M^2}{m^2}} 2(1-4z+z^2)\right]$$

$$\overset{\epsilon\to 0}{=} -\frac{e^2}{(4\pi)^2}\int_0^1 dz\,(1-z)\left[\frac{-4(1-z)^2 m}{(1-z)^2 m^2 + z M^2} + \frac{2(1-4z+z^2)}{((1-z)^2 + z\frac{M^2}{m^2})^2}\, z\frac{2M^2}{m^3}\right]$$

$$\overset{M\to 0}{=} -\frac{e^2}{(4\pi)^2}\int_0^1 dz\,(1-z)\frac{-4(1-z)^2 m}{(1-z)^2 m^2} = \frac{4e^2}{16\pi^2 m}\int_0^1 dz\,(1-z) = \frac{e^2}{8\pi^2 m},$$

where we let $M \to 0$ as it is an IR cutoff for the photon propagator. Further, we took $e(\mu) = e$ to be constant at one-loop order.

$$\frac{\partial \delta_2}{\partial \mu} \overset{M\to 0}{=} -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dz\,\frac{\partial}{\partial \mu}\left[\frac{\Gamma(2-\frac{d}{2})}{((1-z)m)^{4-d}}\left((2-\epsilon)z - \frac{\epsilon}{2}\frac{z}{1-z}(4-2z-\epsilon(1-z))\right)\right]$$

$$= -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dz\,\frac{\overbrace{-2(2-\frac{d}{2})\Gamma(2-\frac{d}{2})}^{1}}{((1-z)m)^{5-d}}\left((2-\epsilon)z - \frac{\epsilon}{2}\frac{z}{1-z}(4-2z-\epsilon(1-z))\right)(1-z)$$

$$= \frac{2e^2}{(4\pi)^2}\int_0^1 dz\,\frac{2z}{m} = \frac{e^2}{8\pi^2 m}$$

$$\delta_3 = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz \frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-\frac{d}{2}}} \left(8z(1-z)\right)$$

$$\frac{\partial \delta_3}{\partial \mu} = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz \frac{\overbrace{2(2-\frac{d}{2})\Gamma(2-\frac{d}{2})}^{1}}{m^{5-d}} \left(8z(1-z)\right) = \frac{2e^2}{(4\pi)^2 m} \int_0^1 dz\, 8z(1-z)$$

$$= \frac{e^2}{\pi^2 m} \int_0^1 dz \left(z - z^2\right) = \frac{e^2}{6\pi^2 m}$$

Thus, putting everyting together, we get

$$\beta_e(\mu) = \mu \frac{\partial}{\partial \mu}\left(-e\delta_1 + e\delta_2 + \frac{e}{2}\delta_3\right)$$

$$= -e\frac{e^2}{8\pi^2} + e\frac{e^2}{8\pi^2} + \frac{e}{2}\frac{e^2}{6\pi^2} = \frac{e^3}{12\pi^2}$$

b) Integrate the solution to the renormalization group equation $\frac{d}{d\log\mu} e(\mu) = \beta_e(\mu)$ to find at one-loop order

$$\alpha(\mu) = \frac{\alpha^*}{1 - \frac{2}{3\pi}\alpha^* \log\left(\frac{\mu}{\mu^*}\right)}, \qquad \alpha = \frac{e^2}{4\pi},$$

Where $\alpha^* = \alpha(\mu^*)$ with μ^* some fixed scale. Plot the behavior of $e(\mu)$. At which scale does e cease to be perturbative?

$$\frac{d}{d\log(\mu)} e(\mu) = \beta_e(\mu) \implies \frac{1}{\beta_e(\mu)} de(\mu) = d\log(\mu)$$

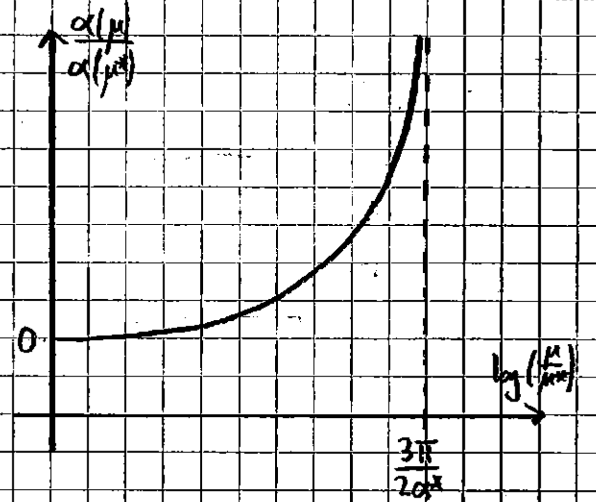
$$\alpha(\mu) = \frac{e^2(\mu)}{4\pi},\ d\alpha(\mu) = \frac{e(\mu)}{2\pi} de(\mu)$$

$$\int_{\mu^*}^{\mu} \frac{1}{\beta_e(\mu')} de(\mu') \overset{\downarrow}{=} \int_{\mu^*}^{\mu} \frac{12\pi^2}{e^3(\mu')}\frac{2\pi}{e(\mu')} d\alpha(\mu') = \int_{\mu^*}^{\mu} \frac{\frac{3}{2}\pi}{\alpha^2(\mu')} d\alpha(\mu')$$

$$= -\frac{3}{2}\pi\left(\frac{1}{\alpha(\mu)} - \frac{1}{\alpha(\mu^*)}\right)$$

$$\int_{\mu^*}^{\mu} d\log(\mu') = \log\left(\frac{\mu}{\mu^*}\right) \overset{!}{=} \frac{3}{2}\pi\left(\frac{1}{\alpha(\mu^*)} - \frac{1}{\alpha(\mu)}\right) \implies \frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu^*)} - \frac{2}{3\pi}\log\left(\frac{\mu}{\mu^*}\right)$$

$$\implies \alpha(\mu) = \frac{\alpha(\mu^*)}{1 - \frac{2}{3\pi}\alpha(\mu^*)\log\left(\frac{\mu}{\mu^*}\right)}$$



Perturbativeness is lost at $e \approx 1$, i.e.

$$\frac{1}{4\pi} \approx \frac{\alpha(\mu^*)}{1 - \frac{2\alpha(\mu^*)}{3\pi}\log\left(\frac{\mu}{\mu^*}\right)}$$

$$\Longrightarrow \mu \approx e^{\frac{3}{2}\pi\left(\frac{1}{\alpha(\mu^*)} - 4\pi\right)}$$

Problem 10.2 (Anomalous dimension)

a) Consider a two-point function $G_2(p;\lambda,\mu)$ of a massless scalar theory and rewrite it as

$$G_2(p;\lambda,\mu) = \frac{i}{p^2} f\left(-\frac{p^2}{\mu^2}\right) \quad (4)$$

with an a priori arbitrary function f. Show that the Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + 2\gamma(\lambda)\right) G_2(p;\lambda,\mu) = 0 \quad (5)$$

can be cast into the form

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda} + 2 - 2\gamma(\lambda)\right) G_2(p;\lambda,\mu) = 0, \quad (6)$$

where $p = \sqrt{-p^2}$.

To arrive at eq. (6), we need a relation between the derivatives of $G_2(p;\lambda,\mu)$ with respect to μ and p. We compute

$$\mu\frac{\partial}{\partial\mu} G_2(p;\lambda,\mu) = \mu\frac{i}{p^2}\frac{2p^2}{\mu^3}\frac{\partial f\left(-\frac{p^2}{\mu^2}\right)}{\partial\left(-\frac{p^2}{\mu^2}\right)} = \frac{2i}{\mu^2} f'\left(-\frac{p^2}{\mu^2}\right)$$

$$p\frac{\partial}{\partial p} G_2(p;\lambda,\mu) = p\frac{-2i}{p^3} f\left(-\frac{p^2}{\mu^2}\right) + p\frac{i}{p^2}\frac{-2p}{\mu^2} f'\left(-\frac{p^2}{\mu^2}\right)$$

$$= -2G_2(p;\lambda,\mu) - \frac{2i}{\mu^2} f'\left(-\frac{p^2}{\mu^2}\right)$$

Thus, we found the relation $\mu\frac{\partial}{\partial\mu}G_2(p;\lambda,\mu) = -p\frac{\partial}{\partial p}G_2(p;\lambda,\mu) - 2G_2(p;\lambda,\mu)$.
Insertion into eq. (5) yields

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda} - 2(1-\gamma)\right)G_2(p;\lambda,\mu) = 0.$$

b) Integrate this equation to find

$$G_2(p;\lambda,\mu) = \frac{1}{p^2}F(\bar\lambda(p;\lambda))\exp\left(2\int_{p'=\mu}^{p} d\left[\lg\frac{p'}{\mu}\right]\gamma(\bar\lambda(p';\lambda))\right), \qquad (7)$$

where $\bar\lambda(p;\lambda)$ is the running coupling, which is defined such that $p\frac{\partial}{\partial p}\bar\lambda(p;\lambda) = \beta(\bar\lambda(p;\lambda))$ with the initial condition $\bar\lambda(\mu;\lambda) = \lambda$, and $F(x)$ is some a priori unknown function.
Hint: Compare eq. (6) with

$$\left(p\frac{d}{dp} + 2(1-\gamma(\lambda^*))\right)G_2(p;\bar\lambda(\tfrac{\mu^2}{p};\lambda^*),\mu)\Big|_{\lambda^*=\bar\lambda(p;\lambda)} = 0 \qquad (8)$$

Note that $\bar\lambda(\frac{\mu^2}{p};x)$ is the inverse of $\bar\lambda(p;x)$, i.e. $\bar\lambda(\frac{\mu^2}{p};\bar\lambda(p;x)) = x$.

We can easily verify that eqs. (6) and (8) are completely equivalent by noting that

$$p\frac{d}{dp}G_2(p;\bar\lambda(\tfrac{\mu^2}{p};\lambda^*),\mu)\Big|_{\lambda^*=\bar\lambda(p;\lambda)} = p\frac{\partial}{\partial p}G_2(p;\bar\lambda(\tfrac{\mu^2}{p};\lambda^*),\mu)\Big|_{\lambda^*=\bar\lambda(p;\lambda)}$$

$$+ p\frac{\partial}{\partial p}\bar\lambda(\tfrac{\mu^2}{p};\lambda^*)\frac{\partial}{\partial\bar\lambda(\frac{\mu^2}{p};\lambda^*)}G_2(p;\bar\lambda(\tfrac{\mu^2}{p};\lambda^*),\mu)\Big|_{\lambda^*=\bar\lambda(p;\lambda)}$$

$$= p\frac{\partial}{\partial p}G_2(p;\bar\lambda(\tfrac{\mu^2}{p};\bar\lambda(p;\lambda)),\mu)$$

$$+ \underbrace{p\frac{\partial}{\partial p}\bar\lambda(\tfrac{\mu^2}{p};\bar\lambda(p;\lambda))}_{\beta(\bar\lambda(\frac{\mu^2}{p};\bar\lambda(p;\lambda)))}\frac{\partial}{\partial\bar\lambda(\frac{\mu^2}{p};\bar\lambda(p;\lambda))}G_2(p;\bar\lambda(\tfrac{\mu^2}{p};\bar\lambda(p;\lambda)),\mu)$$

$$= p\frac{\partial}{\partial p}G_2(p;\lambda,\mu) + \beta(\lambda)\frac{\partial}{\partial\lambda}G_2(p;\lambda,\mu),$$

where we used the hint that $\bar\lambda(\frac{\mu^2}{p};\bar\lambda(p,\lambda)) = \lambda$.

Inserting this equality into eq. (8), we immediately reobtain eq. (6). Thus we may integrate eq. (8) and rest assured that our result for $G_2(p; \lambda, \mu)$ also fulfills eq. (6). Since it is a first order differential equation, we may solve it by separation of variables

$$\int_\mu^p \frac{dG(p'; \bar\lambda(\frac{\mu^2}{p'}; \lambda^*), \mu)}{G(p'; \bar\lambda(\frac{\mu^2}{p'}; \lambda^*), \mu)}\Bigg|_{\lambda^* = \bar\lambda(p; \lambda)} = -2(1-\gamma(\lambda^*)) \int_\mu^p d\ln(p')\Bigg|_{\lambda^* = \bar\lambda(p; \lambda)}$$

$$= \ln(G(p'; \lambda, \mu))\Big|_\mu^p = \ln(G(p; \lambda, \mu)) - \ln\left(\frac{i}{\mu^2} f(-1)\right)$$

$$\Rightarrow G(p; \lambda, \mu) = \frac{i}{\mu^2} f(-1)\, e^{2\int_\mu^p d\ln(p')\gamma(\bar\lambda(p; \lambda))}\, e^{-2\int_\mu^p d\ln(p')}$$

So $\mathcal{F}(\bar\lambda(p; \lambda)) = f(-1)\, e^{-2\int_\mu^p d\ln(p')}$.

c) Argue that in the vicinity of a fixed point λ^* where $\beta(\lambda^*) = 0$ the two-point scales as

$$G_2(p; \lambda^*, \mu) \propto \left(\frac{1}{p^2}\right)^{1-\gamma(\lambda^*)} \qquad (9)$$

and justify from this the term anomalous dimension for γ.

For $\beta(\lambda^*) = 0$, eq. (6) reads at $\lambda = \lambda^*$

$$\frac{d\, G_2(p; \lambda^*, \mu)}{G_2(p; \lambda^*, \mu)} = -2\,(1-\gamma(\lambda^*))\,\frac{dp}{p}$$

Integration gives

$$\ln(G_2(p; \lambda^*, \mu)) = -2\,(1-\gamma(\lambda^*))\,\ln(p) + c = \ln\left(p^{-2(1-\gamma(\lambda^*))}\right) + c$$

Therefore, after exponention we find,

$$G_2(p; \lambda^*, \mu) = e^c\, e^{\ln\left(p^{-2(1-\gamma(\lambda^*))}\right)} \propto \left(\frac{1}{p^2}\right)^{1-\gamma(\lambda^*)},$$

i.e. $G_2(p; \lambda^*, \mu)$ has non-integer - anomalous - dimension of momentum.