

Quantum Field Theory II - Assignment 10

Problem 10.1 (One-loop β -function in QED)

a) Compute the one-loop β -function in QED.

$$\beta_e(\mu) = \mu \frac{\partial}{\partial \mu} \left(-e\delta_1 + e\delta_2 + \frac{e}{2}\delta_3 \right), \quad \text{where } -\mu^2 = 4m^2, \text{ i.e. } \mu \propto m$$

$$\delta_1 = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz (1-z) \left(\frac{\Gamma(2-\frac{d}{2})}{((1-z)^2 m^2 + z M^2)^{2-\frac{d}{2}}} - \frac{1}{2} (2-\epsilon)^2 \right. \\ \left. + \frac{\Gamma(3-\frac{d}{2})}{((1-z)^2 m^2 + z M^2)^{3-\frac{d}{2}}} (2(1-4z+z^2) - \epsilon(1-z)^2) m^2 \right)$$

$$\frac{\partial \delta_1}{\partial \mu} = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz (1-z) \left[\frac{-\frac{d}{2} \Gamma(2-\frac{d}{2})}{((1-z)^2 m^2 + z M^2)^{3-\frac{d}{2}}} 2(1-z)^2 m^2 \frac{1}{2} (2-\epsilon)^2 \right. \\ \left. + \frac{\partial}{\partial \mu} \frac{\Gamma(3-\frac{d}{2})}{((1-z)^2 m^2 + z M^2)^{3-\frac{d}{2}}} 2(1-4z+z^2) \right]$$

$$\stackrel{\epsilon \rightarrow 0}{=} -\frac{e^2}{(4\pi)^2} \int_0^1 dz (1-z) \left[\frac{-4(1-z)^2 m^2}{((1-z)^2 m^2 + z M^2)^2} + \frac{2(1-4z+z^2)}{((1-z)^2 m^2 + z M^2)^2} 2 \frac{M^2}{m^2} \right]$$

$$\stackrel{M \rightarrow 0}{=} -\frac{e^2}{(4\pi)^2} \int_0^1 dz (1-z) \frac{-4(1-z)^2 m^2}{(1-z)^2 m^2} = \frac{4e^2}{16\pi^2 m} \int_0^1 dz (1-z) = \frac{e^2}{8\pi^2 m}$$

where we let $M \rightarrow 0$ as it is an IR cutoff for the photon propagator. Further, we took $e(\mu) = e$ to be constant at one-loop order.

$$\stackrel{M \rightarrow 0}{\frac{\partial \delta_2}{\partial \mu}} = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz \frac{\partial}{\partial \mu} \left[\frac{\Gamma(2-\frac{d}{2})}{((1-z)m^2)^{2-\frac{d}{2}}} \left((2-\epsilon)z - \frac{\epsilon}{2} \frac{z}{1-z} (4-2z-\epsilon(1-z)) \right) \right] \\ = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz \frac{-\frac{d}{2} \Gamma(2-\frac{d}{2})}{((1-z)m^2)^{3-\frac{d}{2}}} \left((2-\epsilon)z - \frac{\epsilon}{2} \frac{z}{1-z} (4-2z-\epsilon(1-z)) \right) (1-z) \\ = \frac{2e^2}{(4\pi)^2} \int_0^1 dz \frac{2z}{m} = \frac{e^2}{8\pi^2 m}$$

$$\delta_3 = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz \frac{\Gamma(\frac{d}{2})}{(\mu^2 z(1-z))^{\frac{d}{2}}} (8z(1-z))$$

$$\frac{\partial S_3}{\partial \mu} = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz \frac{\Gamma(\frac{d}{2})}{m^{5-d}} (8z(1-z)) = -\frac{2e^2}{(4\pi)^{d/2} m} \int_0^1 dz 8z(1-z)$$

$$= -\frac{e^2}{\pi^2 m} \int_0^1 dz (z - z^2) = -\frac{e^2}{6\pi^2 m}$$

Thus, putting everything together, we get

$$\beta_e(\mu) = \mu \frac{\partial}{\partial \mu} \left(-eS_1 + eS_2 + \frac{e}{2} S_3 \right)$$

$$= -e \frac{e^2}{8\pi^2} + e \frac{e^2}{8\pi^2} + \frac{e}{2} \frac{e^2}{6\pi^2} = \frac{e^3}{12\pi^2}$$

b) Integrate the solution to the renormalization group equation

$\frac{d}{d \log \mu} e(\mu) = \beta_e(\mu)$ to find at one-loop order

$$\alpha(\mu) = \frac{\alpha^*}{1 - \frac{2}{3\pi} \alpha^* \log\left(\frac{\mu}{\mu^*}\right)}, \quad \alpha = \frac{e^2}{4\pi}$$

where $\alpha^* = \alpha(\mu^*)$ with μ^* some fixed scale. Plot the behavior of $e(\mu)$. At which scale does e cease to be perturbative?

$$\frac{d}{d \log \mu} e(\mu) = \beta_e(\mu) \Rightarrow \frac{1}{\beta_e(\mu)} de(\mu) = d \log \mu$$

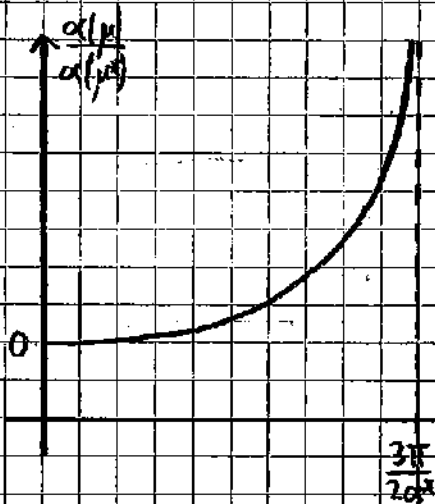
$$\alpha(\mu) = \frac{e(\mu)}{4\pi}, \quad d\alpha(\mu) = \frac{e(\mu)}{2\pi} d \log \mu$$

$$\int_{\mu^*}^{\mu} \frac{1}{\beta_e(\mu)} d\alpha(\mu) = \int_{\mu^*}^{\mu} \frac{12\pi^2}{e^3(\mu)} \frac{2\pi}{e(\mu)} d\alpha(\mu) = \int_{\mu^*}^{\mu} \frac{3\pi}{\alpha^2(\mu)} d\alpha(\mu)$$

$$= -\frac{3}{2} \pi \left(\frac{1}{\alpha(\mu)} - \frac{1}{\alpha(\mu^*)} \right)$$

$$\int_{\mu^*}^{\mu} d \log \mu = \log\left(\frac{\mu}{\mu^*}\right) = -\frac{3}{2} \pi \left(\frac{1}{\alpha(\mu)} - \frac{1}{\alpha(\mu^*)} \right) \Rightarrow \frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu^*)} - \frac{2}{3\pi} \log\left(\frac{\mu}{\mu^*}\right)$$

$$\Rightarrow \alpha(\mu) = \frac{\alpha(\mu^*)}{1 - \frac{2}{3\pi} \alpha(\mu^*) \log\left(\frac{\mu}{\mu^*}\right)}$$



Perturbativeness is lost at $c \approx 1$, i.e.

$$\frac{1}{4\pi} \approx \frac{\alpha(\mu^2)}{1 - \frac{2\alpha(\mu^2)}{3\pi} \log\left(\frac{\mu}{\mu_0}\right)}$$

$$\Rightarrow \mu \approx c^{\frac{3}{2}\pi \left(\frac{1}{\alpha(\mu)} - 4\pi\right)}$$

Problem 10.2 (Anomalous dimension)

a) Consider a two-point function $G_2(p; \lambda, \mu)$ of a massless scalar theory and rewrite it as

$$G_2(p; \lambda, \mu) = \frac{i}{p^2} f\left(-\frac{p^2}{\mu^2}\right) \quad (4)$$

with an a priori arbitrary function f . Show that the Callan-Symanzik equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 2\gamma(\lambda)\right) G_2(p; \lambda, \mu) = 0 \quad (5)$$

can be cast into the form

$$\left(p \frac{\partial}{\partial p} - \beta(\lambda) \frac{\partial}{\partial \lambda} + 2 - 2\gamma(\lambda)\right) G_2(p; \lambda, \mu) = 0, \quad (6)$$

where $p = \sqrt{-p^2}$.

To arrive at eq. (6), we need a relation between the derivatives of $G_2(p; \lambda, \mu)$ with respect to μ and p . We compute

$$\mu \frac{\partial}{\partial \mu} G_2(p; \lambda, \mu) = \mu \frac{i}{p^2} \frac{2p^2}{\mu^3} \frac{\partial f\left(-\frac{p^2}{\mu^2}\right)}{\partial \left(-\frac{p^2}{\mu^2}\right)} = \frac{2i}{\mu^2} f'\left(-\frac{p^2}{\mu^2}\right)$$

$$\begin{aligned} p \frac{\partial}{\partial p} G_2(p; \lambda, \mu) &= p \frac{2i}{p^3} f\left(-\frac{p^2}{\mu^2}\right) + p \frac{i}{p^2} \frac{-2p}{\mu^2} f'\left(-\frac{p^2}{\mu^2}\right) \\ &= -2 G_2(p; \lambda, \mu) - \frac{2i}{\mu^2} f'\left(-\frac{p^2}{\mu^2}\right) \end{aligned}$$

Thus, we found the relation $\mu \frac{\partial}{\partial \mu} G_2(p; \lambda, \mu) = -p \frac{\partial}{\partial p} G_2(p; \lambda, \mu) - 2G_2(p; \lambda, \mu)$

Insertion into eq. (5) yields

$$\left(p \frac{\partial}{\partial p} + \beta(\lambda) \frac{\partial}{\partial \lambda} - 2(1-r) \right) G_2(p; \lambda, \mu) = 0.$$

b) Integrate this equation to find

$$G_2(p; \lambda, \mu) = \frac{1}{p^2} F(\lambda(p; \lambda)) \exp \left(2 \int_{p^2=\mu^2}^p d[\ln \frac{p^2}{\mu^2}] r(\lambda(p; \lambda)) \right), \quad (7)$$

where $\lambda(p; \lambda)$ is the running coupling, which is defined such that $p \frac{\partial}{\partial p} \lambda(p; \lambda) = \beta(\lambda(p; \lambda))$ with the initial condition $\lambda(p; \lambda) = \lambda$, and $F(x)$ is some a priori unknown function.

Hint: Compare eq. (6) with

$$\left(p \frac{\partial}{\partial p} + 2(1-r(x)) \right) G_2(p; \lambda(\frac{\mu^2}{p^2}, x), \mu) \Big|_{x=\lambda(p; \lambda)} = 0. \quad (8)$$

Note that $\lambda(\frac{\mu^2}{p^2}; x)$ is the inverse of $\lambda(p; \lambda)$ i.e. $\lambda(\frac{\mu^2}{p^2}; \lambda(p; \lambda)) = x$.

We can easily verify that eqs. (6) and (8) are completely equivalent by noting that

$$\begin{aligned} p \frac{\partial}{\partial p} G_2(p; \lambda(\frac{\mu^2}{p^2}, x), \mu) \Big|_{x=\lambda(p; \lambda)} &= p \frac{\partial}{\partial p} G_2(p; \lambda(\frac{\mu^2}{p^2}, x), \mu) \Big|_{x=\lambda(p; \lambda)} \\ &+ p \frac{\partial}{\partial p} \lambda(\frac{\mu^2}{p^2}, x) \frac{\partial}{\partial \lambda(\frac{\mu^2}{p^2}, x)} G_2(p; \lambda(\frac{\mu^2}{p^2}, x), \mu) \Big|_{x=\lambda(p; \lambda)} \\ &= p \frac{\partial}{\partial p} G_2(p; \lambda(\frac{\mu^2}{p^2}, \lambda(p; \lambda)), \mu) \\ &+ p \frac{\partial}{\partial p} \lambda(\frac{\mu^2}{p^2}, \lambda(p; \lambda)) \frac{\partial}{\partial \lambda(\frac{\mu^2}{p^2}, \lambda(p; \lambda))} G_2(p; \lambda(\frac{\mu^2}{p^2}, \lambda(p; \lambda)), \mu) \\ &= p \frac{\partial}{\partial p} G_2(p; \lambda, \mu) + \beta(\lambda) \frac{\partial}{\partial \lambda} G_2(p; \lambda, \mu), \end{aligned}$$

where we used the hint that $\lambda(\frac{\mu^2}{p^2}, \lambda(p; \lambda)) = \lambda$.

Inserting this equality into eq. (8), we immediately obtain eq. (6). Thus, we may integrate eq. (8) and rest assured that our result for $G_2(p; \lambda, \mu)$ also fulfills eq. (6). Since it is a first order differential equation, we may solve it by separation of variables

$$\int_{\mu}^p \frac{dG(p; \bar{\lambda}(\frac{p^2}{\mu^2}, \lambda^*), p)}{G(p; \bar{\lambda}(\frac{p^2}{\mu^2}, \lambda^*), p)} \Big|_{\lambda^* = \bar{\lambda}(p; \lambda)} = -2(1 - \gamma(\lambda^*)) \int_{\mu}^p \frac{d \ln(p')}{\lambda^* = \bar{\lambda}(p; \lambda)}$$

$$= \ln(G(p; \lambda, \mu)) \Big|_{\mu}^p = \ln(G(p; \lambda, \mu)) - \ln\left(\frac{1}{\mu^2} f(-1)\right)$$

$$\Rightarrow G(p; \lambda, \mu) = \frac{1}{\mu^2} f(-1) e^{2 \int_{\lambda}^p d \ln(p') \gamma(\bar{\lambda}(p; \lambda))} e^{-2 \int_{\lambda}^p d \ln(p')}$$

$$\text{So } F(\bar{\lambda}(p; \lambda)) = f(-1) e^{-2 \int_{\lambda}^p d \ln(p')}$$

4) Argue that in the vicinity of a fixed point λ^* where $\beta(\lambda^*) = 0$ the two-point scales as

$$G_2(p; \lambda^*, \mu) \propto \left(\frac{1}{p^2}\right)^{1 - \gamma(\lambda^*)} \quad (9)$$

and justify from this the form anomalous dimension for γ .

For $\beta(\lambda^*) = 0$, eq. (6) reads at $\lambda = \lambda^*$

$$\frac{dG_2(p; \lambda^*, p)}{G_2(p; \lambda^*, p)} = -2(1 - \gamma(\lambda^*)) \frac{dp}{p}$$

Integration gives

$$\ln(G_2(p; \lambda^*, \mu)) = -2(1 - \gamma(\lambda^*)) \ln(p) + c = \ln(p^{2(1 - \gamma(\lambda^*))}) + c$$

Therefore, after exponentiation we find,

$$G_2(p; \lambda^*, \mu) = e^c e^{\ln(p^{2(1 - \gamma(\lambda^*))})} \propto \left(\frac{1}{p^2}\right)^{1 - \gamma(\lambda^*)},$$

i.e. $G_2(p; \lambda^*, \mu)$ has non-integer - anomalous - dimension of momentum.