Roland Walz
Janosh Riebesell

1	2	3	Σ
20	20		40

19.06.2015

General Relativity - Exercise Sheet 10

Problem 1 (Quadrupole moment of a rotating bar) [20 points]

Gravitational waves are generated by time-varying quadrupole and higher order moments in the mass distribution. The quadrupole moment Q_{ij} is given by

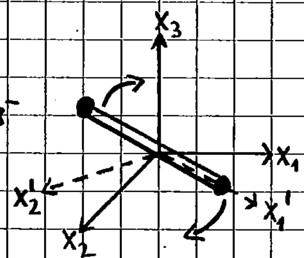
$$Q_{ij} = \int d^3x \, x^i x^j \rho(\vec{x}).$$

a) Consider a solid body dumbbell rotating around the x_3-axis. The coordinate system S' was chosen such that the dumbbell is resting and the quadrupole tensor is diagonal,

$$\theta'^{ij} = \int d^3x' \, x'^i x'^j \rho'(\vec{x}'), \qquad \mathrm{diag}(\underline{\theta}) = (J_1, J_2, J_3).$$

Find a linear transformation $\underline{a}(t)$ such that $x^k = a^k{}_L x'^L$, where the unprimed system S is such that objects surrounding the dumbbell are resting in space, and such that $x_3 = x_3'$.

We need $\underline{\alpha}(t)$ to be time-dependent rotation matrix, i.e. $\underline{\alpha}(t) \in SO(3) \; \forall t$. Preserving the x_3-coordinate, we may write $\alpha(t)$ as



$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos(\tilde{\omega} t) & \sin(\tilde{\omega} t) & 0 \\ -\sin(\tilde{\omega} t) & \cos(\tilde{\omega} t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underline{\alpha}(t) \, \vec{x}'$$

b) Show that $\underline{\theta}(t) = \underline{\alpha}(t) \, \underline{\theta}' \, \underline{\alpha}^T(t)$.

Since $\underline{\alpha}(t) \in SO(3)$ is an orthogonal transformation, the integral measure remains invariant under a change of coordinate system, i.e. $d^3x = d^3x'$. Thus

$$\underline{\alpha}(t) \, \underline{\theta}' \, \underline{\alpha}^T(t) = \int d^3x' \underbrace{\underline{\alpha}(t) \vec{x}'}_{\vec{x}(t)} \underbrace{\vec{x}'^T \underline{\alpha}^T(t)}_{(\alpha(t)\vec{x}')^T = \vec{x}^T(t)} \rho'(\vec{x}') = \int d^3x \, \vec{x}(t) \vec{x}^T(t) \rho(\vec{x}(t)) = \underline{\theta}(t),$$

where $\rho(\vec{x})$ can be thought of as a scalar field, its transformation behavior hence given by $\rho(\vec{x}) \longrightarrow \rho'(\vec{x}) = \rho(\vec{x})$.

c) Explicitly calculate the components of $\underline{\Theta}(t)$.

$$\underline{\Theta}(t) = \underline{\alpha}(t)\, \underline{\Theta}'\, \underline{\alpha}^T(t) = \begin{pmatrix} \cos(\tilde{\omega}t) & \sin(\tilde{\omega}t) & 0 \\ -\sin(\tilde{\omega}t) & \cos(\tilde{\omega}t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix} \begin{pmatrix} \cos(\tilde{\omega}t) & -\sin(\tilde{\omega}t) & 0 \\ \sin(\tilde{\omega}t) & \cos(\tilde{\omega}t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} J_1\cos^2(\tilde{\omega}t) + J_2\sin^2(\tilde{\omega}t) & -(J_1-J_2)\sin(\tilde{\omega}t)\cos(\tilde{\omega}t) & 0 \\ -(J_1-J_2)\sin(\tilde{\omega}t)\cos(\tilde{\omega}t) & J_1\sin^2(\tilde{\omega}t) + J_2\cos^2(\tilde{\omega}t) & 0 \\ 0 & 0 & J_3 \end{pmatrix}$$

d) Show that the components are of the form

$$\Theta^{ij} = c^{ij} + Q^{ij} e^{-2i\tilde{\omega}t} + \bar{Q}^{ij} e^{2i\tilde{\omega}t}, \qquad c^{ij} \in \mathbb{R} \text{ constant}$$

where $Q^{ij} = \frac{1}{4}(J_1 - J_2)(i\sigma_1 + \sigma_3)$ and σ_i are the Pauli matrices with added third row and column filled up by zeros.

Using the trigonometric identities

$$\cos^2(\tilde{\omega}t) = \frac{1}{2}\cos(2\tilde{\omega}t) + \frac{1}{2} = \frac{1}{4}e^{-2i\tilde{\omega}t} + \frac{1}{4}e^{2i\tilde{\omega}t} + \frac{1}{2},$$

$$\sin^2(\tilde{\omega}t) = \frac{1}{2} - \frac{1}{2}\cos(2\tilde{\omega}t) = \frac{1}{2} - \frac{1}{4}e^{-2i\tilde{\omega}t} - \frac{1}{4}e^{2i\tilde{\omega}t},$$

$$\sin(\tilde{\omega}t)\cos(\tilde{\omega}t) = \frac{1}{2}\sin(2\tilde{\omega}t) = \frac{i}{4}e^{-2i\tilde{\omega}t} - \frac{i}{4}e^{2\tilde{\omega}t},$$

we find

$$\underline{\Theta}(t) = \begin{pmatrix} J_2 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix} + \begin{pmatrix} (J_1-J_2)\cos^2(\tilde{\omega}t) & -(J_1-J_2)\sin(\tilde{\omega}t)\cos(\tilde{\omega}t) & 0 \\ -(J_1-J_2)\sin(\tilde{\omega}t)\cos(\tilde{\omega}t) & (J_1-J_2)\sin^2(\tilde{\omega}t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix} + \frac{1}{4}(J_1-J_2)e^{-2i\tilde{\omega}t} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}(J_1-J_2)e^{2i\tilde{\omega}t} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \frac{1}{2}(J_1-J_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underline{c} + \underline{Q}\, e^{-2i\tilde{\omega}t} + \underline{\bar{Q}}\, e^{2i\tilde{\omega}t},$$

where $\underline{C} = \begin{pmatrix} \frac{J_2}{2}(J_1-J_2) & 0 & 0 \\ 0 & \frac{J_2}{2}(J_1-J_2) & 0 \\ 0 & 0 & J_3 \end{pmatrix}$.

e) Rewrite the scalar prefactor in $\underline{Q}$ i.t.o. the moment of inertia w.r.t. the rotational axis J and the ellipticity ϵ,

$$J = J_1 + J_2, \qquad \epsilon = \frac{J_1 - J_2}{J_1 + J_2}.$$

$$\frac{1}{4}(J_1 - J_2) = \frac{1}{4}\epsilon J$$

f) Argue why $\omega = 2\tilde{\omega}$ is the real angular frequency.

Only the time-varying parts of the quadrupole tensor $\underline{Q}$ actually give rise to grav. waves. As shown in part d), these parts oscillate not with angular frequency $\tilde{\omega}$, but with $2\tilde{\omega} =: \omega$. ω is the angular frequency an experimentalist on Earth seeking for gravitational waves would measure, even though the binary system creating those waves rotates with $\tilde{\omega}$. This is obviously because a wave peak is released every time one of the binarics reaches closest approach to the observer, an occurence happening exactly twice per revolution. (The wave peaks need not have equal amplitude.)

g) The power radiated away by gravitational waves is given by

$$P = \frac{128 G\omega^6}{5c^5}\left[\sum_{i,j=1}^{3} |Q^{ij}|^2 - \frac{1}{3}|Q^{ii}|^2\right]$$

Calculate P for a dumbbell in terms of ω, ϵ, and J.

$$\underline{Q} = \frac{1}{4}(J_1 - J_2)(i\sigma_1 + \sigma_3) = \frac{1}{4}(J_1 - J_2)\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P = \frac{128 G\omega^6}{5c^5}\,\frac{4}{16}(J_1 - J_2)^2 = \frac{32 G\omega^6}{5c^5}\,\epsilon^2 J^2$$

Problem 2 (And now for something different) [20 points]

The power radiated out by gravitational waves of a stiff dumbbell is

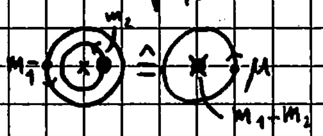
$$P = \frac{32G\omega^6}{5c^5}\,\epsilon^2 J^2.$$

Consider a system of two masses m_1 and m_2 orbiting each other on a spherical orbit, separated by a distance r. In such a system, we may choose the coordinate axis such that

$$J_2 = 0 \qquad J = J_1 = \mu r_1^2 \qquad \epsilon = 1, \qquad \text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

a) Determine the angular frequency ω by requiring that the centrifugal force cancels the gravitational force.

$$F_c = \frac{\mu v^2}{r} = \mu\omega^2 r \overset{!}{=} G\frac{\overbrace{\mu(m_1+m_2)}^{m_1 m_2}}{r^2} = -F_g \implies \omega = \sqrt{\frac{G}{r^3}(m_1+m_2)}$$



b) Express P in terms of m_i and r.

$$P = \frac{32G\omega^6}{5c^5}\epsilon^2 J^2 = \underbrace{\frac{32G^4}{5c^5}}_{\beta}\left(\frac{m_1+m_2}{r^3}\right)^3 \mu^2 r^4 = \beta\,(m_1+m_2)\,m_1^2 m_2^2\,\frac{1}{r^5}$$

c) What is the total energy E of the system?

The potential in this system is given by

$$V_g = -\int F_g\,dr = \int \frac{G m_1 m_2}{r^2}\,dr = -\frac{G m_1 m_2}{r} \propto r^{-1}.$$

Applying the virial theorem $2\langle T_t\rangle = n\langle V_t\rangle$, where $\langle T_t\rangle$ and $\langle V_t\rangle$ are the time-averaged total kinetic and potential energy, and n gives the radial power law of the potential V acting between particles of the system, in this case V_g with $n = -1$, we find

$$E = \langle T_t\rangle + \langle V_t\rangle = -\langle T_t\rangle = -\frac{1}{2}\mu v^2 = -\frac{1}{2}\mu\omega^2 r^2.$$

19.06.2015

d) What is the energy loss per orbit in terms of v and c?

The energy loss per orbit L is the product of power radiated out by gravitational waves P and orbital period $T = \frac{2\pi}{\omega}$.

$$L = P \cdot T = \frac{32 G \omega^6}{5c^5} \epsilon^2 J^2 \frac{2\pi}{\omega} = \frac{64\pi G}{5c^5} \omega^5 \mu^2 r^4 \overset{\omega = \frac{v}{r}}{=} \frac{64\pi G}{5c^5} \frac{\mu^2}{r} \frac{v^5}{c^5}$$

✓

e) Extra: Since $P = -\frac{dE}{dt}$, write down the differential equation for r.

$$\frac{dE}{dt} = \frac{d}{dt}\left(-\frac{1}{2}\mu \omega^2 r^2\right) = \frac{d}{dt}\left(-\frac{1}{2}\mu \frac{G}{r^3}(m_1+m_2) r^2\right) = -\frac{1}{2} G m_1 m_2 \frac{d}{dt}\frac{1}{r}$$

$$= \frac{1}{2} G m_1 m_2 \frac{\dot{r}}{r^2} = -\frac{32 G^4}{5c^5}(m_1+m_2) m_1^2 m_2^2 \frac{1}{r^5} = -P$$

$$\Longrightarrow \dot{r} = -\frac{64 G^3}{5c^5} m_1 m_2 (m_1+m_2) \frac{1}{r^3} =: -\frac{\gamma}{r^3}$$

✓

f) Extra: Substitute $x = \frac{r^4}{r(0)^4}$, and set $\frac{dx}{dt} = -\tau^{-1}$.

$$r = r(0)\, x^{\frac{1}{4}}, \qquad \dot{r} = \frac{r(0)}{4} x^{-\frac{3}{4}} \dot{x} = -\frac{r(0)}{4\tau} x^{-\frac{3}{4}}$$

$$-\frac{r(0)}{4\tau} x^{-\frac{3}{4}} = \frac{-\gamma}{r^3(0)} x^{-\frac{3}{4}} \Longrightarrow \tau = \frac{r^4(0)}{4\gamma}, \quad \text{where } \gamma = \frac{64 G^3}{5c^5} m_1 m_2 (m_1+m_2)$$

✓

g) Extra: What is the solution for x if you require that $x(0) = 1$? Retrieve $r(t)$.

$$x(t) - 1 = x(t) - x(0) = \int_0^t \frac{dx}{dt'} dt' = \int_0^t (-\tau^{-1}) dt' = -\int_0^t \frac{4\gamma}{r^4(0)} dt' = -\frac{4\gamma}{r^4(0)} t$$

$$x(t) = 1 - \frac{4\gamma}{r^4(0)} t$$

✓

Therefore, we find $r(t)$ to be

$$r(t) = \sqrt{r^4(0) - 4\gamma t}$$

Problem 3 (Gravitational wave detectors) [5 points]

Why is it so hard to build gravitational wave detectors? Why have there not been any observations of gravitational waves?

Both questions are answered by the minuteness of the sought-after effect. A gravitational wave observatory attempts to measure 'ripples' in the space-time fabric which cause all objects experiencing these ripples to temporarily change their length. However, based on our knowledge of the binary systems in our galaxy, we may expect to only measure waves of an amplitude sufficient to cause a length scaling by a factor of one in 10^{21} (about one atomic diameter for the distance from sun to Earth). Obviously, it is extremely hard to measure such small effects and separate them from noise.