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General Relativity - Exercise Sheet 11

Problem 1 (h_{0i} again!) [15 points]

The gravitational field $h_{\mu\nu}(\vec{r})$ of a constantly rotating ball of radius R and mass M contains at leading order (in $\frac{v}{c}$) not only a quasi-Newtonian term

$$h_{\mu\mu}(r) = -\frac{2\phi(r)}{c^2}, \qquad \text{where } \phi(r) = \frac{GM}{r},$$

but also a term related to the angular velocity ω,

$$h_{0i}(r) = -\frac{4GMR^2}{5c^3r^3}\,\epsilon_{ijk}\,\omega^j x^k.$$

We will investigate how this perturbation affects geodesics when approximating $dt = d\tau$ and $u = (c, \vec{v})$.

a) Reason why the Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$ can be written as

$$\Gamma^\alpha{}_{\mu\nu} = \frac{\eta^{\alpha\sigma}}{2}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right)$$

in the weak-field limit, in which terms of order $\mathcal{O}(h^2)$ are discarded.
The definition of the Christoffel symbols reads

$$\Gamma^\alpha{}_{\mu\nu} := \frac{g^{\alpha\sigma}}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right).$$

We take $g_{\mu\nu}$ to be Minkowskian metric perturbed by an expansion in $h_{\mu\nu}(\vec{r})$,

$$g_{\mu\nu}(\vec{r}) = \eta_{\mu\nu} \overset{?}{-} h_{\mu\nu}(\vec{r}) + \mathcal{O}(h^2) \approx \eta_{\mu\nu} - h_{\mu\nu}(\vec{r})$$

which we terminate after leading order in $h_{\mu\nu}(\vec{r})$. Since the Minkowskian part of $g_{\mu\nu}(\vec{r})$ is spatially invariant, it drops when taking spatial derivatives, i.e. $\partial_\sigma g_{\mu\nu}(\vec{r}) = \partial_\sigma h_{\mu\nu}(\vec{r})$. If we then multiply these derivatives with $g^{\mu\nu}(\vec{r})$, only the Minkowskian part survives since all combinations of $h^{\mu\nu}(\vec{r})\,\partial_\sigma h_{\mu\nu}(\vec{r})$ are of order $\mathcal{O}(h^2)$ and hence don't contribute in our approximation. Thus,

$$\Gamma^\alpha_{\mu\nu} = \frac{\eta^{\alpha\sigma}}{2}\left(\partial_\mu h_{\nu\sigma}(\vec{r}) + \partial_\nu h_{\mu\sigma}(\vec{r}) - \partial_\sigma h_{\mu\nu}(\vec{r})\right).$$

b) Write down the geodesic equation $\dot{u}^\alpha + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0$ for the metric discussed. Show that it becomes

$$a^i + c^2\Gamma^i_{00} + 2c\,\Gamma^i_{0j} v^j = 0,$$

if we once again discard all terms of order v^2/c^2.

Restricting ourselves to the acceleration's spatial components $a^i = \dot{u}^i$, we get

$$a^i + \Gamma^i_{\mu\nu} u^\mu u^\nu = a^i + \Gamma^i_{00}\underbrace{u^0 u^0}_{c^2} + \Gamma^i_{0j} u^0 v^j + \underbrace{\Gamma^i_{j0}}_{\Gamma^i_{0j}} u^j u^0 + \Gamma^i_{jk}\underbrace{u^j u^k}_{\mathcal{O}(v^2/c^2)}$$

$$= a^i + c^2\Gamma^i_{00} + 2c\,\Gamma^i_{0j} v^j = 0.$$

c) Show that the first term is the expected Newtonian gravity, i.e. $-\partial_i \phi(\vec{r}) = F_{N,i}(\vec{r})$. Show that the second term is $-c(\vec{\nabla}\times\vec{h}\times\vec{v})^i$, where $\vec{h} = (h_{01}, h_{02}, h_{03})^T$. Setting $\vec{\nabla}\times\vec{h} =: \vec{\Omega}$, is it surprising that

$$\vec{F} = m\vec{a} = m\left(-\vec{\nabla}\phi + 2\vec{\Omega}\times\vec{v}\right)$$

is called gravitomagnetic force?

$$c^2\Gamma^i_{00} = \frac{c^2}{2}\eta^{i\sigma}\left(\partial_0 h_{0\sigma}(\vec{r}) + \partial_0 h_{0\sigma}(\vec{r}) - \partial_\sigma h_{00}(\vec{r})\right) = \frac{c^2}{2}\eta^{ii}\left(2\underbrace{\partial_t h_{0i}(\vec{r})}_{0} - \partial_i h_{00}(\vec{r})\right)$$

$$\partial^i\phi \qquad \eta^{ii} = -\delta^{ii}$$

$$= \frac{c^2}{2}\partial_i\left(-\frac{2\phi(\vec{r})}{c^2}\right) = -\partial_i\phi(\vec{r}) =: F_{N,i}(\vec{r})$$

$$2c\,\Gamma^i_{0j} v^j = 2c\,\frac{\eta^{i\sigma}}{2}\left(\underbrace{\partial_0 h_{j\sigma}(\vec{r})}_{0} + \partial_j h_{0\sigma}(\vec{r}) - \partial_\sigma h_{0j}(\vec{r})\right)v^j$$

$$= c\,\eta^{ii}\left(\partial_j h_{0i}(\vec{r}) - \partial_i h_{0j}(\vec{r})\right)v^j = -c\left(\left[\vec{\nabla}\times\vec{h}(\vec{r})\right]\times\vec{v}\right)^i$$

Multiplying the geodesic eq. derived in part b) with a test mass m, we obtain the aforementioned equation of motion.

$$\vec F(\vec r) = m\,\vec a(t) = m\left(\vec\nabla\Phi(\vec r) + \underbrace{c\left[\vec\nabla\times\vec h(\vec r)\right]}_{:=\vec\Omega(\vec r)}\times\vec v\right) = -m\left(-\vec\nabla\Phi(\vec r) + \vec v\times\vec\Omega(\vec r)\right)$$

Except for an overall sign, which can always be absorbed by redefining the fields, this is very reminiscent of the electromagnetic Lorentz force

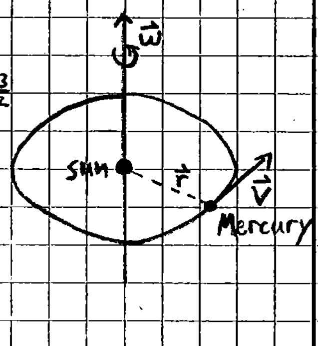
$$F_L(\vec r) = e\left(-\vec\nabla\phi(\vec r) + \vec v\times\vec B(\vec r)\right),$$

justifying the name of gravitomagnetic force (except maybe for the fact that there is nothing magnetic about it).

d) Compare both contributions for the effect the sun and its rotation have on the innermost planet, Mercury, under the assumption that Mercury's orbital plane is perpendicular to the sun's rotational axis.

$$\left|-\vec\nabla\Phi(\vec r)\right| = \left|-\vec\nabla\frac{GM}{r}\right| = \left|\frac{GM}{r^2}\vec e_r\right| = \frac{GM}{r^2}$$

With $\vec h(\vec r) = -\frac{4GMR^2}{5c^3r^3}\vec\omega\times\vec r$, where $\vec r$ points at Mercury's current position, we get for the second contribution



$$\vec\nabla\times\left(\vec\omega\times\frac{\vec r}{r^3}\right) = \vec\nabla\times\frac{1}{r^3}\begin{pmatrix}\omega_2 r_3-\omega_3 r_2\\ \omega_3 r_1-\omega_1 r_3\\ \omega_1 r_2-\omega_2 r_1\end{pmatrix}, \qquad \frac{1}{r^3} = \left(r_1^2+r_2^2+r_3^2\right)^{-\frac{3}{2}}$$

$$= \begin{pmatrix}\partial_2(\omega_1 r_2-\omega_2 r_1)/r^3 - \partial_3(\omega_3 r_1-\omega_1 r_3)/r^3\\ \partial_3(\omega_2 r_3-\omega_3 r_2)/r^3 - \partial_1(\omega_1 r_2-\omega_2 r_1)/r^3\\ \partial_1(\omega_3 r_1-\omega_1 r_3)/r^3 - \partial_2(\omega_2 r_3-\omega_3 r_2)/r^3\end{pmatrix}$$

$$= \begin{pmatrix}\omega_1/r^3 - 3r_2(\omega_1 r_2-\omega_2 r_1)/r^5 + \omega_1/r^3 + 3r_3(\omega_3 r_1-\omega_1 r_3)/r^5\\ \omega_2/r^3 - 3r_3(\omega_2 r_3-\omega_3 r_2)/r^5 + \omega_2/r^3 + 3r_1(\omega_1 r_2-\omega_2 r_1)/r^5\\ \omega_3/r^3 - 3r_1(\omega_3 r_1-\omega_1 r_3)/r^5 + \omega_3/r^3 + 3r_2(\omega_2 r_3-\omega_3 r_2)/r^5\end{pmatrix}$$

$$= \frac{2}{r^3}\vec\omega - \frac{3}{r^5}\vec r\times(\vec\omega\times\vec r) = \frac{2}{r^3}\vec\omega - \frac{3}{r^5}\Big(\vec\omega\underbrace{(\vec r\cdot\vec r)}_{r^2} - \vec r\underbrace{(\vec r\cdot\vec\omega)}_{0}\Big) = -\frac{1}{r^3}\vec\omega$$

$$\left|-c\left[\vec\nabla\times\vec h(\vec r)\right]\times\vec v\right| = \left|\frac{4GMR^2}{5c^2}\left[\underbrace{\vec\nabla\times\left(\vec\omega\times\frac{\vec r}{r^3}\right)}_{-\frac{1}{r^3}\vec\omega}\right]\times\vec v\right| = \left|-\frac{4GMR^2}{5c^2r^3}\,\vec\omega\times\vec v\right|$$

$$= \left|-\frac{4GMR^2}{5c^2r^3}\,|\vec\omega|\,|\vec v|\,\vec e_r\right| = \frac{4GMR^2}{5c^2r^3}\,\omega v,$$

Where ω is the angular velocity of the sun and v is the orbital speed of Mercury. We denote all parameters describing the sun with an index $\odot$, all those of Mercury with an index M. Using $\omega = 2\pi f = \frac{2\pi}{T}$ and $v = \omega r = \frac{2\pi r}{T}$, we get the following comparison

$$\frac{|-\vec\nabla\phi(\vec r)|}{|-c[\vec\nabla\times\vec h(\vec r)]\times\vec v_M|} = \frac{\frac{GM_\odot}{r_M^2}}{\frac{4GMR_\odot^2}{5c^2r_M^3}\omega_\odot v_M} = \frac{5c^2r_M}{4R_\odot^2\,\omega_\odot v_M} = \frac{5c^2r_M}{4R_\odot^2\frac{2\pi}{T_\odot}\frac{2\pi r_M}{T_M}} = \frac{5c^2T_\odot T_M}{16\pi^2R_\odot^2}$$

$$\approx 9.538\cdot10^{10},$$

when inserting $T_\odot = 25\,d$, $T_M = 88\,d$, $R_\odot^2 = 7\cdot10^8\,m$.

Problem 2 (Particles reacting to gravitational waves) [15 points]

Consider a gravitational wave in the 1-2 plane and the transverse-traceless gauge amplitude tensor

$$\underline{A} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & a & b & 0\\ 0 & b & -a & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

a) What does the time-dependent part of the metric $g_{\mu\nu}(\vec x,t) = \eta_{\mu\nu} + h_{\mu\nu}(\vec x,t)$ describing a gravitational wave look like? How many components of x does h depend upon?

Since $A_{\mu\nu}$ is traceless, $\bar h_{\mu\nu} = h_{\mu\nu} = A_{\mu\nu}\exp(\pm ik_\sigma x^\sigma)$.

Therefore, the time dependence of the metric goes like $g_{\mu\nu}(\vec x,t) \propto e^{\pm ik_0x^0} = e^{\pm i\omega t}$.

We may always transform our coordinates such that $k = (\omega, 0, 0, k_z)$. Thus, $g_{\mu\nu}$ depends only on two x-components.

b) Considering the geodesic eq. from the above exercise, what can we say about the spatial acceleration of a particle a^i at $\tau = 0$ if we require that the particles were resting at $\tau = 0$? What are the conclusions about $x^i(\tau)$?

If $v^i(\tau=0) = 0$, then $2c\Gamma^i_{0j} v^j = 0$. Looking at $A_{\mu\nu}$, we see that $c^2\Gamma^i_{00}$ too is zero. Therefore, $a^i = 0$ and hence $v^i = 0 \quad \forall \tau$. Consequently, $x^i(\tau) = \text{const}$, meaning particles don't change their position. ✓

c) We split up the line element according to

$$ds^2 = c^2 dt^2 - dl^2 - (dx^3)^2,$$

then $dl^2 = (\delta_{ij} - h_{ij}(t))\, dx^i dx^j$, where $i,j \in \{1,2\}$.
Write h i.t.o. the Pauli matrices σ.

$$h_{\mu\nu}(\vec{x},t) = A_{\mu\nu}\, e^{\pm i(\omega t - k_z z)}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu$$

$$= c^2 dt^2 - (\delta_{ij} - h_{ij})\, dx^i dx^j - (d^3x)^2$$

$$\underline{h} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} e^{\pm i(\omega t - k_z z)} = (b\sigma_1 + a\sigma_3)\, e^{\pm i(\omega t - k_z z)}$$

The required Pauli matrices for the above expression of $\underline{h}$ are both real and traceless. ✓

d) We imagine a circle of Particles P in the 1-2-plane with Radius R, enabling the description of particle positions by an angle φ.
The physical distance from the centre $\hat{R}$ in physical coordinates x_p^j

$$\hat{R}^2 = (\delta_{ij} - h_{ij}(t))\, x_p^i\, x_p^j$$

Define those physical coordinates on the plane and then calculate $\hat{R}^2$ for i) the '+'- and ii) the 'x'-modes, i.e. $a = h$, $b = 0$ and $a = 0$, $b = h$, where h is the scalar amplitude.
We set $\vec{x}_p = \begin{pmatrix} x_p^1 \\ x_p^2 \end{pmatrix} = R \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$, with $R \in \mathbb{R}$ constant.
In the 1-2-plane $z = 0$ and so $e^{\pm i(\omega t - k_z z)} = e^{\pm i\omega t}$.
To obtain a physical, i.e. real, distance $\hat{R}$ from the centre of the circle, we superposition $h_{ij} = A_{ij}(e^{+i\omega t} + e^{-i\omega t}) = 2A_{ij}\cos(\omega t)$

i) $\hat{R}^2 = \delta_{ij} x_p^i x_p^j - h_{ij}(t)\, x_p^i x_p^j = R^2\cos^2\varphi + R^2\sin^2\varphi$

$- 2h\cos(\omega t) R^2 \cos^2\varphi + 2h\cos(\omega t) R^2 \sin^2\varphi$

$= R^2(1 - 2h\cos(\omega t)\cos(2\varphi))$

ii) $\hat{R}^2 = \delta_{ij} x_p^i x_p^j - h_{ij}(t)\, x_p^i x_p^j = R^2 - 2h\cos(\omega t) R^2 \cos\varphi \sin\varphi$

$- 2h\cos(\omega t) R^2 \cos\varphi \sin\varphi$

$= R^2(1 - 4h\cos(\omega t)\cos\varphi\sin\varphi) = R^2(1 - 2h\cos(\omega t)\sin(2\varphi))$

Problem 3 (Gravitational wave induced ellipticity) [10 points]

The '+'-mode derived in exercise 2 reads

not clear! -1

$$\hat{R}^2 = R(1 - 2h\cos(2\varphi)\cos(\omega t)).$$

We consider weak fields for which $h \ll 1$.

a) What needs to be fulfilled for maximum displacement?

$$\cos(2\varphi)\cos(\omega t) = \pm 1 \iff \varphi = m\pi,\ t = \frac{n\pi}{\omega},\ m,n \in \mathbb{Z}$$

Then $\hat{R} = R(1 \pm 2h)$.

b) Write $\hat{R}^2$ in the form of an ellipse, i.e.

$$\frac{x^2}{\epsilon_1^2} + \frac{y^2}{\epsilon_2^2} = 1.$$

What are the ϵ_i in terms of h?

$$\hat{R}^2 = R^2(1 - 2h\cos(2\varphi)\cos(\omega t))$$

$$= R^2(1 - 2h(\cos^2\varphi - \sin^2\varphi)\cos(\omega t))$$

$$= R^2 - 2h(x^2 - y^2)\cos(\omega t) = x^2 + y^2$$

$$(1 + 2h\cos(\omega t))\frac{x^2}{R^2} + (1 - 2h\cos(\omega t))\frac{y^2}{R^2} = 1$$

Therefore, $\epsilon_1^2 = \frac{R^2}{1 + 2h\cos(\omega t)}$ and $\epsilon_2^2 = \frac{R^2}{1 - 2h\cos(\omega t)}$. ✓

c) ϵ_1 and ϵ_2 are the semiminor and semimajor axes of an ellipse. Calculate the eccentricity ϵ induced by a gravitational '+'-polarised wave with scalar amplitude h.

✓

$$\epsilon = \sqrt{1 - \frac{\epsilon_2^2}{\epsilon_1^2}} = \left(1 - \frac{1 - 2h\cos(\omega t)}{1 + 2h\cos(\omega t)}\right)^{\frac{1}{2}}$$

Problem 4 (Extra: Wave equations) [5 points]

One of the decisive differences between gravity and the electromagnetic force is that the former knows only one charge while the latter has two.

Interchanging two opposite charges, as in an oscillating electric dipole, changes the electric field distribution. That's why electric dipoles radiate off energy in the form of waves.

Gravitational mass on the other hand, comes in only one sign. Interchanging two masses therefore does not result in a change of the gravitational field distribution. Thus, at least a varying quadrupole is necessary to perturb the fields and emit gravitational waves. ✓

This gives the graviton spin 2, while the photon only has spin 1. Examples for spin 0 waves with just one polarisation state are the Higgs boson or even accoustical sound. ✓