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03.07.2015

General Relativity - Exercise Sheet 12

Problem 1 (Lie derivative) [15 points]

The Lie derivative $(\mathcal{L}_a v)^\mu$ was established via a coordinate transform of $\bar{v}^\mu(\bar{x})$ to $v^\mu(x)$ for the r.h.s., and an expansion of $\bar{v}^\mu$ around $\bar{x}$ for the l.h.s.

a) Derive the Lie derivative of a (1,1)-tensor $Z^\mu{}_\nu$ in the same fashion.

In general, the Lie derivative of an arbitrary tensor $T \in \mathcal{T}^r_s$ of rank (r,s) with respect to a vector field a on a manifold M is defined as

$$\mathcal{L}_a T = \lim_{t \to 0} \frac{\gamma_t^* T - T}{t},$$

where γ_t is the flow of a.

Evidently, the Lie derivative of rank (r,s)-tensor is itself again a rank (r,s)-tensor.

To find an expression for the components of a Lie derivative of a (1,1)-tensor $\underline{Z}$, we apply the Lie derivative to the tensor product $\underline{Z} \otimes dx^\mu \otimes \partial_\nu$, in which the set $\{\partial_\mu\}$ forms a coordinate basis of the vector field a and the set $\{dx^\mu\}$ is its dual basis.

$$\mathcal{L}_a(\underline{Z} \otimes dx^\mu \otimes \partial_\nu) \overset{\text{Leibniz rule}}{=} (\mathcal{L}_a \underline{Z}) \otimes dx^\mu \otimes \partial_\nu + \underline{Z} \otimes (\mathcal{L}_a dx^\mu) \otimes \partial_\nu$$
$$+ \underline{Z} \otimes dx^\mu \otimes (\mathcal{L}_a \partial_\nu)$$

A full contraction of the above yields

$$\underbrace{\mathcal{L}_a Z^\mu{}_\nu}_{a^\rho \partial_\rho Z^\mu{}_\nu} = (\mathcal{L}_a \underline{Z})^\mu{}_\nu + \underbrace{\underline{Z}(\partial_\rho a^\mu dx^\rho, \partial_\nu)}_{(\partial_\rho a^\mu) Z^\rho{}_\nu} - \underbrace{\underline{Z}(dx^\mu, \partial_\nu a^\rho \partial_\rho)}_{(\partial_\nu a^\rho) Z^\mu{}_\rho}$$

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Solving for the components of the Lie derivative of $\underline{Z}$, i.e. $(\mathcal{L}_a \underline{Z})^\mu{}_\nu$, we arrive at

$$(\mathcal{L}_a \underline{Z})^\mu{}_\nu = a^\rho \partial_\rho Z^\mu{}_\nu - (\partial_\rho a^\mu) Z^\rho{}_\nu + (\partial_\nu a^\rho) Z^\mu{}_\rho$$

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b) Compare the result of part a) to the one given in the script for the (0,2)-tensor $g_{\mu\nu}$. What is the difference?

The Lie derivative of the (0,2)-metric-tensor $g_{\mu\nu}$, as taken from the script, reads

$$\begin{aligned}(\mathcal{L}_a g)_{\mu\nu} &= a^\rho \partial_\rho g_{\mu\nu} + (\partial_\nu a^\rho) g_{\mu\rho} + (\partial_\mu a^\rho) g_{\rho\nu} \\ &= a^\rho \partial_\rho g_{\mu\nu} - a^\rho \Gamma^\sigma{}_{\rho\mu} g_{\sigma\nu} - a^\rho \Gamma^\sigma{}_{\rho\nu} g_{\mu\sigma} + a^\rho \Gamma^\sigma{}_{\rho\mu} g_{\sigma\nu} \\ &\quad + a^\rho \Gamma^\sigma{}_{\rho\nu} g_{\mu\sigma} + (\partial_\nu a^\sigma) g_{\mu\sigma} + (\partial_\mu a^\sigma) g_{\sigma\nu} \\ &= a^\rho \underbrace{\nabla_\rho g_{\mu\nu}}_{0} + (\nabla_\nu a^\sigma) g_{\mu\sigma} + (\nabla_\mu a^\sigma) g_{\sigma\nu} \\ &= (\nabla_\nu a^\sigma) g_{\mu\sigma} + (\nabla_\mu a^\sigma) g_{\sigma\nu} \\ &= \nabla_\nu a_\mu + \nabla_\mu a_\nu, \qquad (= 0 \text{ if } a_\mu \text{ is a Killing vector field})\end{aligned}$$

where $\nabla_\rho g_{\mu\nu} = 0$ because we are free to choose an affine connection and vanishing torsion such that a parallel transport operation preserving lengths and angles is ensured.

This condition, which we may not impose for an arbitrary rank (1,1)-tensor such as $\underline{Z}$ is the reason why we were able to derive the much simpler expression for the Lie derivative of the metric given above.

03.07.2015

Problem 2 (Killing vectors) [15 points]

The Lie-derivative of the metric $g_{\mu\nu}$ is given by

$$(\mathcal{L}_a g)_{\mu\nu} = g_{\mu\lambda}\nabla_\nu a^\lambda + g_{\lambda\nu}\nabla_\mu a^\lambda = \nabla_\nu a_\mu + \nabla_\mu a_\nu,$$

where ∇_μ are covariant derivatives. For a Killing vector ξ, we have $(\mathcal{L}_\xi g)_{\mu\nu} = 0$.

a) For the 2-sphere, $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. Show that the Killing vector field is related to the angular momentum.

From the line element $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$, where $\mu,\nu \in \{1,2\}$ with $x^1 = \theta$, $x^2 = \varphi$, we extract the following metric

$$g_{11} = 1 = g^{11}, \quad g_{22} = \sin^2\theta = \frac{1}{g^{22}}, \quad g_{12} = g^{12} = g_{21} = g^{21} = 0.$$

Using $\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$, we find that only three of the eight Christoffel symbols of the 2-sphere are non-vanishing.

$$\Gamma^1{}_{22} = -\sin\theta\cos\theta, \qquad (\Gamma^1{}_{11} = \Gamma^1{}_{12} = \Gamma^1{}_{21} = 0)$$

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot\theta, \qquad (\Gamma^2{}_{11} = \Gamma^2{}_{22} = 0)$$

Killing's equations $(\mathcal{L}_\xi g)_{\mu\nu} = 0$ then provide the following determining conditions for a Killing vector ξ:

i) $0 = (\mathcal{L}_\xi g)_{11} = \nabla_1 \xi_1 + \nabla_1 \xi_1 = 2(\partial_1 \xi_1 - \underbrace{\Gamma^\rho{}_{11}}_{0\ \forall\rho} \xi_\rho) = 2\,\partial_\theta \xi_\theta$

ii) $0 = (\mathcal{L}_\xi g)_{22} = \nabla_2 \xi_2 + \nabla_2 \xi_2 = 2(\partial_2 \xi_2 - \Gamma^\rho{}_{22} \xi_\rho) = 2(\partial_\phi \xi_\phi + \sin\theta\cos\theta\, \xi_\theta)$

iii) $0 = (\mathcal{L}_\xi g)_{12} = (\mathcal{L}_\xi g)_{21} = \nabla_1 \xi_2 + \nabla_2 \xi_1 = \partial_1 \xi_2 - \Gamma^\rho{}_{12} \xi_\rho + \partial_2 \xi_1 - \Gamma^\rho{}_{21} \xi_\rho$

$= \partial_\theta \xi_\phi + \partial_\phi \xi_\theta - 2\cot\theta\, \xi_\phi$

From condition i), we gather that $K_\theta(\theta,\phi) = K_\theta(\phi)$. Thus, if we can construct a differential equation for $K_\theta(\phi)$, it should be easy to give a solution. We differentiate iii) w.r.t. ϕ to get

$$\partial_\theta \partial_\phi \xi_\phi + \partial_\phi^2 \xi_\theta - 2\cot\theta\, \partial_\phi \xi_\phi .$$

Inserting ii), i.e. $\partial_\phi \xi_\phi = -\sin\theta\cos\theta\, \xi_\theta$, we obtain

$$\partial_\theta(-\sin\theta\cos\theta\,\xi_\theta) + \partial_\phi^2 \xi_\theta - 2\cot\theta(-\sin\theta\cos\theta\,\xi_\theta)$$

$$= -\cos^2\theta\,\xi_\theta + \sin^2\theta\,\xi_\theta + \partial_\phi^2\xi_\theta + 2\cos^2\theta\,\xi_\theta$$

$$= \partial_\phi^2 \xi_\theta + \xi_\theta ,$$

which is exactly the differential equation describing the harmonic oscillator. It's most general solution is

$$\xi_\theta(\phi) = c_1 \sin\phi + c_2\cos\phi , \qquad \text{with } c_1, c_2 \text{ constant.}$$

$\xi_\phi(\theta,\phi)$ is still undetermined, so we insert $\xi_\theta(\phi)$ into ii),

$$\partial_\phi \xi_\phi = -\sin\theta\cos\theta\,(c_1\sin\phi + c_2\cos\phi)$$

and integrate to obtain

$$\xi_\phi(\theta,\phi) = \sin\theta\cos\theta\,(c_1\cos(\phi) - c_2\sin\phi) + f(\theta),$$

where $f(\theta)$ is an, as yet, unknown function solving the homogeneous differential equation $\partial_\phi \xi_\phi = 0$. It can, however, be determined by inserting $\xi_\theta(\phi)$ and $\xi_\phi(\theta,\phi)$ into iii):

$$\partial_\theta \xi_\phi + \partial_\phi \xi_\theta - 2\cot\theta\,\xi_\phi = \partial_\theta\big(\sin\theta\cos\theta\,(c_1\cos\phi - c_2\sin\phi) + f(\theta)\big)$$

$$+ \partial_\phi(c_1\sin\phi + c_2\cos\phi) - 2\cot\theta\big(\sin\theta\cos\theta\,(c_1\cos\phi - c_2\sin\phi) + f(\theta)\big)$$

$$= (\cos^2\theta - \sin^2\theta)(c_1\cos\phi - c_2\sin\phi) + \frac{\partial f(\theta)}{\partial\theta} + c_1\cos\phi - c_2\sin\phi$$

$$-2\cos^2\theta\,(c_1\cos\phi - c_2\sin\phi) - 2\cot\theta\, f(\theta) = \frac{\partial f(\theta)}{\partial\theta} - 2\cot\theta\, f(\theta) = 0$$

Division by $\sin^2\theta$ and reverse application of the product rule gives

$$\frac{1}{\sin^2\theta}\frac{\partial f(\theta)}{\partial\theta} - \underbrace{2\frac{\cos\theta}{\sin^3\theta}}_{\partial_\theta \frac{1}{\sin^2\theta}} f(\theta) = \partial_\theta\left(\frac{f(\theta)}{\sin^2\theta}\right) = 0.$$

Thus, $f(\theta)$ must be of the form $f(\theta) = c_3\sin^2\theta$ with c_3 constant. Therefore, the general form for the Killing vectors of the 2-sphere is

$$\xi_\theta = c_1\sin\varphi + c_2\cos\varphi,$$

$$\xi_\phi = \sin\theta\cos\theta\,(c_1\cos\varphi - c_2\sin\varphi) + c_3\sin^2\theta,$$

with tree independent constants c_1, c_2, c_3 yielding three linearly independent Killing vectors. Expanding the Killing vectors in an orthogonal basis of the tangent space of S^2, we may write them as

$$\xi = \xi_\theta\,\partial_\theta + \xi_\varphi\,\partial_\phi,$$

where ∂_θ and ∂_ϕ span the tangent space. Then we find for a particular choice of c_1, c_2, and c_3, that

$$\xi_1 = \sin\phi\,\partial_\theta + \sin\theta\cos\theta\cos\phi\,\partial_\phi, \qquad \text{for } c_1 = 1,\ c_2 = c_3 = 0,$$

$$\xi_2 = -\cos\phi\,\partial_\theta + \sin\theta\cos\theta\sin\phi\,\partial_\phi, \qquad \text{for } c_2 = -1,\ c_1 = c_3 = 0,$$

$$\xi_3 = -\sin^2\theta\,\partial_\phi, \qquad \text{for } c_3 = -1,\ c_1 = c_2 = 0.$$

This is already a nice result. However, to see the relation to the contravariant angular momentum vector, we need to compute the contravariant form of the Killing vectors via contraction with the metric. We find

$$\xi_1 = g^{11} \sin\phi \, \partial_\theta + g^{22} \sin\theta \cos\theta \cos\phi \, \partial_\phi$$

$$= \sin\phi \, \partial_\theta + \cot\theta \cos\phi \, \partial_\phi ,$$

$$\xi_2 = -g^{11} \cos\phi \, \partial_\theta + g^{22} \sin\theta \cos\theta \sin\phi \, \partial_\phi$$

$$= -\cos\phi \, \partial_\theta + \cot\theta \cos\phi \, \partial_\phi ,$$

$$\xi_3 = -g^{22} \sin^2\theta \, \partial_\phi = -\partial_\phi .$$

Since in spherical coordinates, the components of the angular momentum operator are given by

$$L_x = i\hbar (\sin\phi \, \partial_\theta + \cot\theta \cos\phi \, \partial_\phi),$$

$$L_y = i\hbar (-\cos\phi \, \partial_\theta + \cot\theta \sin\phi \, \partial_\phi),$$

$$L_z = -i\hbar \, \partial_\phi ,$$

we found that for our particular choice of c_1, c_2, c_3, the three contravariant Killing vectors are, up to a factor of $i\hbar$, identical to the components of angular momentum.

b) Show that for a Killing vector ξ and the stress-energy-tensor $T^{\mu\nu}$, we can define a current $J^\mu = \xi_\nu T^{\mu\nu}$ which has a vanishing covariant derivative, i.e. $\nabla_\mu J^\mu = 0$.

$$\nabla_\mu J^\mu = \nabla_\mu (\xi_\nu T^{\mu\nu}) = \underbrace{(\nabla_\mu \xi_\nu)}_{-\nabla_\nu \xi_\mu} T^{\mu\nu} + \xi_\nu \underbrace{\nabla_\mu T^{\mu\nu}}_{0}$$

$$= -(\nabla_\nu \xi_\mu) T^{\nu\mu} - \xi_\mu \nabla_\nu T^{\nu\mu} = -\nabla_\nu (\xi_\mu T^{\nu\mu}) = -\nabla_\nu J^\nu$$

Problem 4 (Extra: Torsion and Cosmology) [5 points]

a) What would happen to Killing vectors if $\Gamma^{\alpha}_{\mu\nu} \neq \Gamma^{\alpha}_{\nu\mu}$?

We would no longer have $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$. The Lie derivative of the metric w.r.t. a Killing vector field would not vanish.

b) Is there a timelike Killing vector in FLRW-cosmologies?

A metric can only admit a timelike Killing vector $\frac{\partial}{\partial \tau}$ if it is stationary, i.e. if there exist coordinates in which the metric is independent of time. (This would mean observers travelling along integral curves of $\frac{d}{d\tau}$ do not notice any change in the universe.)
Since the metric of FLRW-cosmologies is given by

$$\mathrm{diag}(g) = (1, -a(t), -a(t), -a(t))$$

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in cartesian coordinates, these do not, except for the special case of $a(t) = a$ constant, contain timelike Killing vector fields.