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General Relativity - Practice Sheet 1

Problem 1 (Noether's theorem)

Given the Lagrangian

$$\mathcal{L}(r,\dot r,\varphi,\dot\varphi,t) = \frac{m}{2}(\dot r,\dot\varphi)\begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix}\begin{pmatrix}\dot r\\ \dot\varphi\end{pmatrix} - \frac{\alpha}{r} = \frac{m}{2}(\dot r^2 + r^2\dot\varphi^2) - \frac{\alpha}{r},$$

a) find $\frac{d}{dt}\mathcal{L}$? What does the result imply?

$$\frac{d}{dt}\mathcal{L} \overset{*_1}{=} \frac{\partial\mathcal{L}}{\partial r}\frac{dr}{dt} + \frac{\partial\mathcal{L}}{\partial\dot r}\frac{d\dot r}{dt} + \frac{\partial\mathcal{L}}{\partial\varphi}\frac{d\varphi}{dt} + \frac{\partial\mathcal{L}}{\partial\dot\varphi}\frac{d\dot\varphi}{dt} + \underbrace{\frac{\partial\mathcal{L}}{\partial t}}_{0}\frac{dt}{dt}$$

*₁ chain rule

$$= \frac{\partial\mathcal{L}}{\partial r}\dot r + \frac{\partial\mathcal{L}}{\partial\dot r}\ddot r + \frac{\partial\mathcal{L}}{\partial\varphi}\dot\varphi + \frac{\partial\mathcal{L}}{\partial\dot\varphi}\ddot\varphi$$

$$\overset{*_2}{=} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot r}\dot r + \frac{\partial\mathcal{L}}{\partial\dot r}\ddot r + \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\varphi}\dot\varphi + \frac{\partial\mathcal{L}}{\partial\dot\varphi}\ddot\varphi$$

$$\overset{*_3}{=} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot r}\dot r + \frac{\partial\mathcal{L}}{\partial\dot\varphi}\dot\varphi\right) = \frac{d}{dt}\left(p_r\dot r + p_\varphi\dot\varphi\right)$$

*₂ here we restrict the generality of our final cond. by imposing the Euler-Lagrange equations of motion

*₃ inverse product rule

From this result, which we obtained only because our Lagrangian is not explicitly time-dependent, we can deduce that the total energy of any system described by a Lagrangian with this property is conserved over time, i.e.

$$\frac{d}{dt}H = \frac{d}{dt}\left(p_r\dot r + p_\varphi\dot\varphi - \mathcal{L}\right) = \frac{d}{dt}\left(p_r\dot r + p_\varphi\dot\varphi\right) - \frac{d}{dt}\left(p_r\dot r + p_\varphi\dot\varphi\right) = 0$$

b) find $\frac{d}{d\varphi}\mathcal{L}$. What does the result imply?

$$\frac{d}{d\varphi}\mathcal{L} \overset{*_1}{=} \frac{\partial\mathcal{L}}{\partial r}\underbrace{\frac{dr}{d\varphi}}_{0} + \frac{\partial\mathcal{L}}{\partial\dot r}\underbrace{\frac{d\dot r}{d\varphi}}_{0} + \underbrace{\frac{\partial\mathcal{L}}{\partial\varphi}}_{0}\frac{d\varphi}{d\varphi} + \frac{\partial\mathcal{L}}{\partial\dot\varphi}\underbrace{\frac{d\dot\varphi}{d\varphi}}_{0} + \frac{\partial\mathcal{L}}{\partial t}\underbrace{\frac{dt}{d\varphi}}_{0} = 0$$

$\frac{d\mathcal{L}}{d\varphi} = 0$ means that φ is a cyclic coordinate and that $p_\varphi = \frac{\partial\mathcal{L}}{\partial\dot\varphi}$, the canonically conjugate momentum of φ, also known as the angular momentum, is conserved over time if one imposes the equations of motion, i.e.

$$\frac{d}{dt}L_\varphi = \frac{d}{dt}p_\varphi = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\varphi} \overset{!}{=} \frac{\partial\mathcal{L}}{\partial\varphi} = 0$$

Problem 2 (Virial theorem)

The physical system in this exercise is taken to be contained in a finite volume and to possess a finite amount of energy.
A function $f(x)$ is said to b homogeneous of degree k iff $f(\alpha x) = \alpha^k f(x)$.
Euler's homogeneous function theorem states that for a homogeneous function $f(\vec{x})$

$$\vec{x} \cdot \vec{\nabla} f(\vec{x}) = k f(\vec{x}).$$

It is easy to see that the function $T(\vec{v}) = \frac{m}{2}\vec{v}^2$ is homogeneous of degree 2. Therefore, by the aforementioned theorem,

$$\frac{\partial T}{\partial v_i} v_i = p_i v_i = 2T(\vec{v}).$$

We rewrite the left-hand term as

$$p_i v_i = \frac{d}{dt}(p_i x_i) - \dot{p}_i x_i.$$

If we average over time,

$$\langle f \rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t dt' \, f(t'),$$

we get $2\langle T \rangle = \left\langle \frac{d}{dt}(p_i x_i) - \dot{p}_i x_i \right\rangle = \langle -\dot{p}_i x_i \rangle$.

a) Why does the first term vanish?

$$\left\langle \frac{d}{dt}(p_i x_i) \right\rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t dt' \frac{d}{dt'}(p_i x_i) = \lim_{t\to\infty} \frac{1}{t}(p_i x_i)\Big|_0^t = 0$$

b) Rewrite the remaining term as a derivative of the potential $V(x)$.

Since the Lagrangian is generally of the form $\mathcal{L} = T(\dot{q}_i) - V(q_i)$, where in this case $q_i = x_i$, $\dot{q}_i = \dot{x}_i = v_i$, we can express $\dot{p}_i$ in terms of the derivative of $V(x)$.

$$\dot{p}_i = \frac{d}{dt} p_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \overset{\text{e.o.m.}}{=} \frac{\partial \mathcal{L}}{\partial x_i} = -\frac{\partial V(x_i)}{\partial x_i} \implies \langle -\dot{p}_i x_i \rangle = \left\langle \frac{\partial V(x_i)}{\partial x_i} x_i \right\rangle = \langle \vec{x}\,\vec{\nabla} V(x) \rangle = \langle k\, V(x) \rangle,$$

where the last step requires $V(\vec{x})$ to be a homogeneous function of order k.

c) Apply Euler's theorem to the potential, as it is a homogeneous function of degree k as well as $T(\vec{v})$.
See answer to part b) of this problem.

d) What does this imply for $\langle T \rangle$ and $\langle V \rangle$ in gravitational/electrostatic potentials $(V(r) \propto r^{-1})$ and harmonic potentials $(V(r) \propto r^2)$?
Taking our result from 2.b) and plugging it into the expression for $2\langle T \rangle$, we get the Virial theorem

$$2\langle T \rangle = \langle kV \rangle, \quad k \in \mathbb{Z}$$

In the case of gravitational and electrostatic potentials, we have $k=-1$ so that the Virial theorem yields

$$2\langle T \rangle = -\langle V \rangle,$$

whereas for harmonic potentials, $k=2$ so that

$$\langle T \rangle = \langle V \rangle.$$

Problem 3 (Gravitational 'Opticks')
In general relativity, light rays follow null geodesics, meaning that their paths can be 'bent' by the presence of mass. The deflection angle α in general relativity is given by

$$\alpha = \frac{4GM}{c^2 r_0}, \tag{1}$$

where r_0 is the impact parameter (expressing how far away from the centre of the mass the rays are passing). Let us try and recreate this effect with Newtonian gravity.
a) Calculate the escape velocity of a particle from a given potential $V = -G\frac{mM}{r}$ from the simple argument that the initial kinetic energy

T_i has to be the same as the negtive potential energy $-V_i$, so that the particle's total energy $E = T_i + V_i$ is exactly zero by the time it has completely escaped the potential.

$$T_i = \frac{m}{2} v_{esc}^2 = +G\frac{mM}{r} = -V_i \implies v_{esc} = \sqrt{\frac{2GM}{r}}$$

For $v_{esc} = c$, we obtain the famous Schwarzschild radius R_s

$$R_s = \frac{2GM}{c^2}.$$

b) Let's consider the movement of a massive body (later to be called light ray) in a Newtonian gravitational potential again,

$$F = m\frac{d^2 r}{d^2 t} = +G\frac{mM}{r^2}.$$

As we're interested in the body reaching us (instead of being bound by the potential), we are looking for hyperbolae as solutions to the above e.o.m. These shall be parametrised by

$$r(\varphi) = \frac{r_0(1+e)}{1+e\cos\varphi}, \qquad \frac{d\varphi}{dt} = \frac{1}{r^2}\sqrt{GM\, r_0(1+e)}, \qquad (3)$$

where the impact parameter r_0 is the distance of closest approach during the passage, and e is the eccentricity. Writing the vector $\vec{r}$ as

$$\vec{r} = r\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix},$$

calculate the velocity $\vec{v} = \frac{d\vec{r}}{dt}$ and its square.

$$\frac{d}{dt}\vec{e}_r = \frac{d}{dt}\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix} = \begin{pmatrix}-\sin\varphi \\ \cos\varphi\end{pmatrix}\frac{d\varphi}{dt} = \vec{e}_\varphi\,\dot{\varphi}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\vec{e}_r) = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\dot{\varphi}\,\vec{e}_\varphi$$

$$\vec{v}^2 = \dot{r}^2\underbrace{\vec{e}_r^{\,2}}_{1} + 2r\dot{r}\dot{\varphi}\underbrace{\vec{e}_r\vec{e}_\varphi}_{0} + r^2\dot{\varphi}^2\underbrace{\vec{e}_\varphi^{\,2}}_{1} = \dot{r}^2 + r^2\dot{\varphi}^2$$

c) Let's consider $r \longrightarrow \infty$, which is when the massive body (light ray) reaches us. What condition can you then infer from eq. (3) for the final angle φ_∞. If we define

$$\varphi_\infty = \frac{\pi}{2} + \delta,$$

what relation do we get for $\sin\delta$ and e.

Solving eq. (3) for φ_∞ when performing $r \to \infty$, we get

$$\varphi_\infty = \arccos\left(-\frac{1}{e}\right)$$

Using our definition of φ_∞, we find

$$\varphi_\infty = \frac{\pi}{2} + \delta = \arccos\left(-\frac{1}{e}\right) \Longrightarrow \sin\delta = \sin\left(\arccos\left(-\frac{1}{e}\right) - \frac{\pi}{2}\right) = \cos\left(\arccos\left(-\frac{1}{e}\right)\right) = -\frac{1}{e}$$

Apparently, the final deflection angle $\alpha = 2\delta = 2\arcsin\left(-\frac{1}{e}\right) = -2\arcsin\left(\frac{1}{e}\right)$ depends only on the eccentricity e of the hyperbola, and it in turn on the impact parameter ξ_1.

d) Take your result from part b) and set $\vec{v}^2 = c^2$. Then calculate the deflection angle $\alpha = 2\delta$ for the Newtonian case. For this, we can assume small angles δ. How does the fraction compare to unity? Can you simplify further? Are there any parallels to the general relativistic deflection angle? Calculate the Newtonian deflection angle of the sun at its surface

For $\vec{v}^2 = c^2$, we have $c^2 = \dot{r}^2 + r^2\dot{\varphi}^2$. At the point of closest approach, we can use that $\dot{r} = 0$, since r has a minimum at closest approach, and that $r = \xi_1$. Therefore, at this point, it holds that

$$c^2 = \xi_1^2\dot{\varphi}^2 = \xi_1^2\,\frac{GM\xi_1(1+e)}{\xi_1^4} = \frac{1}{\xi_1}GM(1+e),$$

or, when solved for e, $e = \frac{c^2\xi_1}{GM} - 1$.

Inserting for example mass and radius of the sun for M and r_1, which are of the order $M_\odot \approx 2\cdot 10^{30}$ kg and $R_\odot = 7\cdot 10^8$ m, we find that $\frac{c^2 r_1}{GM}$ is much larger than one, so that we may write

$$e \approx \frac{c^2 r_1}{GM}$$

For small angles δ, we can estimate $\sin\delta \approx \delta$ so that our Newtonian formula for the deflection angle α_N reads

$$\alpha_N = 2\delta \approx -\frac{2}{e} \approx -\frac{2GM}{c^2 r_1}$$

We find the Newtonian deflection angle to be smaller by a factor of (-)2. At the sun's surface, it takes the value

$$\alpha_{N,\odot} = -\frac{2GM_\odot}{c^2 R_\odot} = -4.25\cdot 10^{-6}.$$

Problem 4 (Constant constants)

In the lecture, we discussed the four 'most fundamental constant of modern physics, c, G, $\hbar$, and k_B.

a) Which change would be most noticeable if one of those were suddenly to be multiplied by a factor of two?

Impossible to tell.

b) Which of these would you consider 'least fundamental'?

None.