String Theory – Solution to Assignment 1

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1 Differential Geometry for General Relativity

Consider the line element of a 2-sphere of radius a,

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = a^2\big[\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\big]. \tag{1}$$

The metric $g_{\mu\nu}$ encodes all information on the geometry of a manifold. From it one can determine all those geometric quantities that are relevant for general relativity, namely

The metric Choosing $x^1 = \theta$ and $x^2 = \varphi$, read off the matrix $g_{\mu\nu}$.

The Christoffel symbols The Christoffel symbols are defined as

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}\left(\frac{\partial g_{\mu\kappa}}{\partial x^\nu} + \frac{\partial g_{\nu\kappa}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right). \tag{2}$$

They enter the covariant derivatives $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda}V^\lambda$, where the correction term with the Christoffel symbol ensures that the covariant derivative indeed transforms covariantly under arbitrary coordinate transformations $x^\mu \to x'^\mu(x^\nu)$, i.e.

$$\nabla_\mu V^\nu = \partial_\mu V^\nu \to \big(\nabla_\mu V^\nu = \partial_\mu V^\nu\big)' = \frac{\partial x^\lambda}{\partial x'^\mu}\,\frac{\partial x'^\nu}{\partial x^\rho}\,\nabla_\lambda V^\rho, \tag{3}$$

without second derivatives in the coordinates. Compute the non-vanishing Christoffel symbols for the 2-sphere (Hint: $\Gamma^\kappa{}_{\lambda\mu} = \Gamma^\kappa{}_{\mu\lambda}$).

The Riemann tensor The Riemann curvature tensor has the form

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\kappa{}_{\lambda\nu} - \partial_\nu\Gamma^\kappa{}_{\lambda\mu} + \Gamma^\eta{}_{\lambda\nu}\Gamma^\kappa{}_{\eta\mu} - \Gamma^\eta{}_{\lambda\mu}\Gamma^\kappa{}_{\nu\eta}. \tag{4}$$

Calculate the non-vanishing components of the Riemann tensor for the 2-sphere (Hint: $R^\kappa{}_{\lambda\mu\nu} = -R^\kappa{}_{\lambda\nu\mu}$).

Remark: The Riemann tensor measures the curvature of a space, for instance by quantifying the non-commutativity of the covariant derivatives,

$$\big[\nabla_\mu, \nabla_\nu\big]V^\kappa = R^\kappa{}_{\lambda\mu\nu}V^\lambda. \tag{5}$$

A space with vanishing Riemann tensor is flat, i.e. the metric can be brought to the standard Minkowskian (or Euclidean) form by means of a coordinate transformation.

The Ricci tensor The Ricci tensor is defined as

$$R_{\mu\nu} = R^\kappa{}_{\mu\kappa\nu}. \tag{6}$$

Calculate the Ricci tensor for the 2-sphere.

The scalar curvature The scalar curvature is given by

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu}. \tag{7}$$

Calculate the scalar curvature of the 2-sphere. How does it behave in the limit $a \to \infty$? Interpret this behavior.

The Einstein tensor The Einstein tensor appears in the field equation of general relativity and it relates the curvature of space-time to the matter distribution,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{8}$$

where G denotes Newton's constant, $T_{\mu\nu}$ is the energy-momentum tensor, and $G_{\mu\nu}$ denotes the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu}. \tag{9}$$

Calculate the Einstein tensor for the 2-sphere.

The metric For $x^1 = \theta$ and $x^2 = \varphi$, the equation

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = a^2\left[\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right] \tag{10}$$

implies

$$\boldsymbol{g} = a^2\begin{pmatrix}1 & 0\\ 0 & \sin^2(\theta)\end{pmatrix}, \qquad \text{and hence} \quad \boldsymbol{g}^{-1} = \frac{1}{a^2}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{\sin^2}(\theta)\end{pmatrix}. \tag{11}$$

The Christoffel symbols Since the Christoffel symbols carry three coordinate indices and we have $d = 2$ dimensions (θ, φ), there are $d^3 = 8$ Christoffel symbols in total. However, due to the symmetry in the lower two indices, those with lower indices 12 and 21 are equal both for an upper index of 1 and 2, so only $8 - 2 = 6$ of those symbols are independent. We calculate each of those in turn. Since the only nonvanishing metric derivative is $\frac{\partial g_{22}}{\partial x^1} = 2\sin(\theta)\cos(\theta)$, all but ${\Gamma^1}_{22}$, ${\Gamma^2}_{12}$, and ${\Gamma^2}_{21}$ can immediately be seen to vanish:

$$\begin{aligned}
{\Gamma^1}_{11} &= \frac{1}{2}g^{1\kappa}\left(\frac{\partial g_{1\kappa}}{\partial x^1} + \frac{\partial g_{1\kappa}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^\kappa}\right) = 0,\\
{\Gamma^1}_{12} &= \frac{1}{2}g^{1\kappa}\left(\frac{\partial g_{1\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^\kappa}\right) = \frac{1}{2}g^{12}\frac{\partial g_{22}}{\partial x^1} = 0 = {\Gamma^1}_{21},\\
{\Gamma^1}_{22} &= \frac{1}{2}g^{1\kappa}\left(\frac{\partial g_{2\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^\kappa}\right)\\
&= -\frac{1}{2}g^{11}\frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2a^2}\frac{\partial\left[a^2\sin^2(\theta)\right]}{\partial\theta} = -\sin(\theta)\cos(\theta),\\
{\Gamma^2}_{11} &= \frac{1}{2}g^{2\kappa}\left(\frac{\partial g_{1\kappa}}{\partial x^1} + \frac{\partial g_{1\kappa}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^\kappa}\right) = 0,\\
{\Gamma^2}_{12} &= \frac{1}{2}g^{2\kappa}\left(\frac{\partial g_{1\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^\kappa}\right)\\
&= \frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial x^1} = \frac{1}{2a^2\sin^2(\theta)}\frac{\partial\left[a^2\sin^2(\theta)\right]}{\partial\theta} = \cot(\theta) = {\Gamma^2}_{21},\\
{\Gamma^2}_{22} &= \frac{1}{2}g^{2\kappa}\left(\frac{\partial g_{2\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^\kappa}\right) = 0.
\end{aligned} \tag{12}$$

The Riemann tensor The Riemann tensor has four indices. So for $d = 2$ dimensions, the tensor contains a total of $d^4 = 16$ components. Due to the antisymmetry ${R^\kappa}_{\lambda\mu\nu} = -{R^\kappa}_{\lambda\nu\mu}$ of the last two indices, there is only $\frac{d}{2}(d-1) = 1$ independent combination of those two indices, leaving

the Riemann tensor with $d^2 \cdot \frac{d}{2}(d-1) = 4$ independent components. In particular, all eight entries where the last two indices are equal must be zero, i.e.

$$\begin{aligned} R^1{}_{111} = R^1{}_{211} = R^2{}_{111} = R^2{}_{211} = 0, \\ R^1{}_{122} = R^1{}_{222} = R^2{}_{122} = R^2{}_{222} = 0. \end{aligned} \tag{13}$$

The remaining eight components are potentially nonzero, but form four pairs of two whose members differ only in sign. These we calculate by hand:

$$\begin{aligned} R^1{}_{112} &= -R^1{}_{121} = \partial_1 \Gamma^1{}_{12} - \partial_2 \Gamma^1{}_{11} + \Gamma^\eta{}_{12} \Gamma^1{}_{\eta 1} - \Gamma^\eta{}_{11} \Gamma^1{}_{2\eta} = 0, \\ R^1{}_{212} &= -R^1{}_{221} = \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^1{}_{21} + \Gamma^\eta{}_{22} \Gamma^1{}_{\eta 1} - \Gamma^\eta{}_{21} \Gamma^1{}_{2\eta} = \partial_1 \Gamma^1{}_{22} - \Gamma^2{}_{21} \Gamma^1{}_{22} \\ &= \partial_\theta [-\sin(\theta)\cos(\theta)] - \cot(\theta) \cdot [-\sin(\theta)\cos(\theta)] \\ &= -\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta) = \sin^2(\theta), \\ R^2{}_{112} &= -R^2{}_{121} = \partial_1 \Gamma^2{}_{12} - \partial_2 \Gamma^2{}_{11} + \Gamma^\eta{}_{12} \Gamma^2{}_{\eta 1} - \Gamma^\eta{}_{11} \Gamma^2{}_{2\eta} = \partial_1 \Gamma^2{}_{12} + \Gamma^2{}_{12} \Gamma^2{}_{21} \\ &= \partial_\theta \cot(\theta) + \cot(\theta) \cdot \cot(\theta) = -1 - \cot^2(\theta) + \cot^2(\theta) = -1, \\ R^2{}_{212} &= -R^2{}_{221} = \partial_1 \Gamma^2{}_{22} - \partial_2 \Gamma^2{}_{21} + \Gamma^\eta{}_{22} \Gamma^2{}_{\eta 1} - \Gamma^\eta{}_{21} \Gamma^2{}_{2\eta} = 0. \end{aligned} \tag{14}$$

We found four nonvanishing components. The remark given in the exercise that a space with vanishing Riemann tensor is flat is in fact an "iff"-statement, i.e. a nonvanishing Riemann tensor implies that space is curved. We have therefore proven the unremarkable statement that the 2-sphere is curved.

The Ricci tensor The $d^2 = 4$ components of the Ricci tensor of the 2-sphere are given by

$$\begin{aligned} R_{11} = R^\kappa{}_{1\kappa 1} = R^1{}_{111} + R^2{}_{121} = 1, \\ R_{12} = R^\kappa{}_{1\kappa 2} = R^1{}_{112} + R^2{}_{122} = 0, \\ R_{21} = R^\kappa{}_{2\kappa 1} = R^1{}_{211} + R^2{}_{221} = 0, \\ R_{22} = R^\kappa{}_{2\kappa 2} = R^1{}_{212} + R^2{}_{222} = \sin^2(\theta). \end{aligned} \tag{15}$$

The scalar curvature In the case of the 2-sphere, $\mathcal{R}$ takes the very simple and memorable form,

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = g^{11} R_{11} + g^{22} R_{22} = \frac{1}{a^2} \cdot 1 + \frac{1}{a^2 \sin^2(\theta)} \cdot \sin^2(\theta) = \frac{2}{a^2}. \tag{16}$$

As expected, in the limit $a \to \infty$, we have

$$\lim_{a \to \infty} \mathcal{R} = 0, \tag{17}$$

i.e. a sphere of infinite radius has vanishing curvature.

The Einstein tensor We give the Einstein tensor of the 2-sphere not component-wise, but in covariant matrix form:

$$\boldsymbol{G} = \boldsymbol{R} - \frac{1}{2} \mathcal{R} \boldsymbol{g} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} - \frac{1}{2} \frac{2}{a^2} a^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{18}$$

According to the Einstein equation,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{19}$$

a vanishing Einstein tensor requires a trivial matter distribution, $T_{\mu\nu} = 0$.

2 Transformation of tensors and tensor densities

Consider the coordinate change

$$x^\mu \to x'^\mu \equiv x^{\mu'}. \tag{20}$$

The associated transformation matrix and its inverse are

$$P^\mu{}_{\nu'} = \frac{\partial x^\mu}{\partial x'^\nu}, \qquad \text{and} \qquad P^{\mu'}{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}, \tag{21}$$

respectively. Recall that a tensor of type, say $T^\mu{}_\nu$ transforms under eq. (20) as

$$T^\mu{}_\nu \to T^{\mu'}{}_{\nu'} = P^{\mu'}{}_\alpha P^\beta{}_{\nu'} T^\alpha{}_\beta. \tag{22}$$

A tensor density $\tilde{T}^\mu{}_\nu$ of weight w is defined by the transformation behavior

$$\tilde{T}^\mu{}_\nu \to \tilde{T}^{\mu'}{}_{\nu'} = J^w P^{\mu'}{}_\alpha P^\beta{}_{\nu'} \tilde{T}^\alpha{}_\beta. \tag{23}$$

(and obvious generalisations for general types of tensor densities), where $J = \det(\boldsymbol{P})$.

a) Given the tensor $S_{\mu\nu}$, convince yourself that $\sqrt{\det(\boldsymbol{S})}$ is a scalar density of weight 1.

b) Consider now fields of tensors and tensor densities, e.g. $T^\mu{}_\nu(x)$. Locally, i.e. infinitesimally, the transformation of eq. (20) can be parametrized as $x'^\mu = x^\mu - \epsilon^\mu(x)$. Show the following infinitesimal variations for a scalar field $\Phi(x)$, the metric $g_{\mu\nu}(x)$ and the associated metric density $\sqrt{-\det(\boldsymbol{g})}$:
   i) $\delta\Phi = \epsilon^\mu \partial_\mu \Phi$,
   ii) $\delta g_{\mu\nu} = \epsilon^\lambda \partial_\lambda g_{\mu\nu} + \big(\partial_\mu \epsilon^\lambda\big) g_{\lambda\nu} + \big(\partial_\nu \epsilon^\lambda\big) g_{\mu\lambda} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$,
   iii) $\delta\sqrt{-\det(\boldsymbol{g})} = \partial_\lambda \Big[\epsilon^\lambda \sqrt{-\det(\boldsymbol{g})}\Big]$,

   where the second equality in ii) is true for the metric connection satisfying $\nabla_\lambda g_{\mu\nu} = 0$.

   Hint: For a scalar field the transformed object is defined via the relation $\Phi'(x') = \Phi(x)$.

a) Since $S_{\mu\nu}$ is said to be a tensor, we know it transforms as

$$S_{\mu\nu} \to S_{\mu'\nu'} = P^\alpha{}_{\mu'} P^\beta{}_{\nu'} S_{\alpha\beta}. \tag{24}$$

   or in matrix notation

$$\boldsymbol{S} \to \boldsymbol{S}' = \boldsymbol{P}^2_{x\to x'} \boldsymbol{S}. \tag{25}$$

   Using that the determinant of a product of matrices is the product of the determinants, we have

$$\begin{aligned} \sqrt{\det(\boldsymbol{S})} \to \sqrt{\det(\boldsymbol{S}')} &= \sqrt{\det\big(\boldsymbol{P}^2_{x\to x'} \boldsymbol{S}\big)} = \sqrt{\big[\det(\boldsymbol{P}_{x\to x'})\big]^2 \det(\boldsymbol{S})} \\ &= \det(\boldsymbol{P}_{x\to x'}) \sqrt{\det(\boldsymbol{S})} = J^1_{x\to x'} \sqrt{\det(\boldsymbol{S})}, \end{aligned} \tag{26}$$

   and thus $\sqrt{\det(\boldsymbol{S})}$ is a tensor density of weight 1.

b) We now derive the infinitesimal transformation behavior of a scalar field as well as the metric and the square root of its negated determinant under the transformation $x'^\mu = x^\mu - \epsilon^\mu(x')$.
   i) Using $x^\mu = x'^\mu + \epsilon^\mu(x')$, the variation of a **scalar field** follows from the hint that $\Phi'(x') = \Phi(x)$ together with a simple Taylor expansion of $\Phi(x)$ around x',

$$\begin{aligned}\Phi(x) \to \Phi'(x') \stackrel{!}{=} \Phi(x) &= \Phi(x' + \epsilon(x')) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \Phi(x' + \epsilon(x'))}{\partial^n (x'^{\mu} + \epsilon^{\mu}(x'))} \bigg|_{x'^{\mu} + \epsilon^{\mu}(x') = x'^{\mu}} (x'^{\mu} + \epsilon^{\mu}(x') - x'^{\mu})^n \\ &= \Phi(x') + \epsilon^{\mu}(x') \partial_{\mu'} \Phi(x') + \mathcal{O}\big[\epsilon^2(x')\big].\end{aligned} \tag{27}$$

From this transformation law it follows that

$$\Phi'(x) = \Phi(x) + \epsilon^{\mu}(x) \partial_{\mu} \Phi(x) + \mathcal{O}\big[\epsilon^2(x)\big], \tag{28}$$

and hence the variation $\delta\Phi(x)$ of the scalar field $\Phi(x)$ given by the difference of the transformed and the original field reads

$$\delta\Phi(x) = \Phi'(x) - \Phi(x) = \epsilon^{\mu}(x) \partial_{\mu} \Phi(x) + \mathcal{O}\big[\epsilon^2(x)\big]. \tag{29}$$

ii) For the **metric** $g_{\mu\nu}(x)$, we know that it strictly follows the transformational behavior (22) of a tensor (field). Therefore,

$$g_{\mu\nu}(x) \to g_{\mu'\nu'}(x') = P^{\alpha}{}_{\mu'} P^{\beta}{}_{\nu'} g_{\alpha\beta}(x). \tag{30}$$

Now, all we have to do is expand the expression on the right to first order in $\epsilon^{\mu}(x)$. This can be done by inserting our transformation $x^{\mu} = x'^{\mu} + \epsilon^{\mu}(x')$ into the definition of the transformation matrix $P^{\mu}{}_{\nu'}$

$$P^{\alpha}{}_{\mu'} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} = \frac{\partial (x'^{\alpha} + \epsilon^{\alpha}(x'))}{\partial x'^{\mu}} = \delta^{\alpha}{}_{\mu} + \partial_{\mu'} \epsilon^{\alpha}(x'). \tag{31}$$

For the metric $g_{\alpha\beta}(x)$, we do a Taylor expansion,

$$\begin{aligned}g_{\alpha\beta}(x) &= g_{\alpha\beta}(x' + \epsilon(x')) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n g_{\alpha\beta}(x' + \epsilon(x'))}{\partial^n (x'^{\gamma} + \epsilon^{\gamma}(x'))} \bigg|_{x'^{\gamma} + \epsilon^{\gamma}(x') = x'^{\gamma}} (x'^{\gamma} + \epsilon^{\gamma}(x') - x'^{\gamma})^n \\ &= g_{\alpha\beta}(x') + \epsilon^{\gamma}(x') \partial_{\gamma'} g_{\alpha\beta}(x') + \mathcal{O}\big[\epsilon^2(x')\big].\end{aligned} \tag{32}$$

Inserting eq. (31) and the Taylor expansion into eq. (30), we get

$$\begin{aligned}g_{\mu'\nu'}(x') &= \big[\delta^{\alpha}{}_{\mu} + \partial_{\mu'} \epsilon^{\alpha}(x')\big] \big[\delta^{\beta}{}_{\nu} + \partial_{\nu'} \epsilon^{\beta}(x')\big] \big[g_{\alpha\beta}(x') + \epsilon^{\gamma}(x') \partial_{\gamma'} g_{\alpha\beta}(x') + \mathcal{O}\big[\epsilon^2(x')\big]\big] \\ &= \delta^{\alpha}{}_{\mu} \delta^{\beta}{}_{\nu} g_{\alpha\beta}(x') + \big[\partial_{\mu'} \epsilon^{\alpha}(x')\big] \delta^{\beta}{}_{\nu} g_{\alpha\beta}(x') + \delta^{\alpha}{}_{\mu} \big[\partial_{\nu'} \epsilon^{\beta}(x')\big] g_{\alpha\beta}(x') \\ &\quad + \delta^{\alpha}{}_{\mu} \delta^{\beta}{}_{\nu} \epsilon^{\gamma}(x') \partial_{\gamma'} g_{\alpha\beta}(x') + \mathcal{O}\big[\epsilon^2(x')\big] \\ &= g_{\mu\nu}(x') + g_{\alpha\nu}(x') \partial_{\mu'} \epsilon^{\alpha}(x') + g_{\mu\beta}(x') \partial_{\nu'} \epsilon^{\beta}(x') + \epsilon^{\gamma}(x') \partial_{\gamma'} g_{\mu\nu}(x') + \mathcal{O}\big[\epsilon^2(x')\big].\end{aligned} \tag{33}$$

From this we infer that

$$g_{\mu'\nu'}(x) = g_{\mu\nu}(x) + g_{\alpha\nu}(x) \partial_{\mu} \epsilon^{\alpha}(x) + g_{\mu\beta}(x) \partial_{\nu} \epsilon^{\beta}(x) + \epsilon^{\gamma}(x) \partial_{\gamma} g_{\mu\nu}(x) + \mathcal{O}\big[\epsilon^2(x)\big], \tag{34}$$

and hence

$$\begin{aligned}\delta g_{\mu\nu}(x) &= g_{\mu'\nu'}(x) - g_{\mu\nu}(x) \\ &= g_{\alpha\nu}(x) \partial_{\mu} \epsilon^{\alpha}(x) + g_{\mu\beta}(x) \partial_{\nu} \epsilon^{\beta}(x) + \epsilon^{\gamma}(x) \partial_{\gamma} g_{\mu\nu}(x) + \mathcal{O}\big[\epsilon^2(x)\big].\end{aligned} \tag{35}$$

To see that the second equality in ii) holds, we simply calculate

$$\nabla_{\mu} \epsilon_{\nu} + \nabla_{\nu} \epsilon_{\mu} = \nabla_{\mu} (g_{\alpha\nu} \epsilon^{\alpha}) + \nabla_{\nu} \big(g_{\mu\beta} \epsilon^{\beta}\big) = g_{\alpha\nu} \nabla_{\mu} \epsilon^{\alpha} + g_{\mu\beta} \nabla_{\nu} \epsilon^{\beta} + \epsilon^{\gamma} \underbrace{\nabla_{\gamma} g_{\mu\nu}}_{0} \tag{36}$$

$$
\begin{aligned}
&= g_{\alpha\nu}\big(\partial_\mu\epsilon^\alpha + \Gamma^\alpha{}_{\mu\delta}\epsilon^\delta\big) + g_{\mu\beta}\big(\partial_\nu\epsilon^\beta + \Gamma^\beta{}_{\nu\zeta}\epsilon^\zeta\big)\\
&\quad +\epsilon^\gamma\big(\partial_\gamma g_{\mu\nu} - \Gamma^\alpha{}_{\mu\gamma}g_{\alpha\nu} - \Gamma^\beta{}_{\nu\gamma}g_{\mu\beta}\big)\\
&= g_{\alpha\nu}\partial_\mu\epsilon^\alpha + g_{\mu\beta}\partial_\nu\epsilon^\beta + \epsilon^\gamma\partial_\gamma g_{\mu\nu}\\
&\quad +g_{\alpha\nu}\Gamma^\alpha{}_{\mu\delta}\epsilon^\delta - \epsilon^\gamma\Gamma^\alpha{}_{\mu\gamma}g_{\alpha\nu} + g_{\mu\beta}\Gamma^\beta{}_{\nu\zeta}\epsilon^\zeta - \epsilon^\gamma\Gamma^\beta{}_{\nu\gamma}g_{\mu\beta}\\
&= g_{\alpha\nu}\partial_\mu\epsilon^\alpha + g_{\mu\beta}\partial_\nu\epsilon^\beta + \epsilon^\gamma\partial_\gamma g_{\mu\nu} = \delta g_{\mu\nu} - \mathcal{O}[\epsilon^2].
\end{aligned}
\tag{36}
$$

iii) By part a), we have

$$
\sqrt{-\det[\boldsymbol{g}(x)]} \to \sqrt{-\det[\boldsymbol{g}'(x')]} = J_{x\to x'}\sqrt{-\det[\boldsymbol{g}(x)]} = J_{x\to x'}g(x), \tag{37}
$$

where to save on writing, we introduced the shorthand notation $\sqrt{-\det[\boldsymbol{g}(x)]} = g(x)$. To find the infinitesimal variation $\delta\sqrt{-\det[\boldsymbol{g}(x)]} = \delta g(x)$, we expand eq. (37) to first order in $\epsilon(x)$,

$$
\begin{aligned}
g'(x') &= J_{x\to x'}g(x) = \det(\boldsymbol{P}_{x\to x'})g(x'+\epsilon(x'))\\
&= \underbrace{\det[\mathbb{1} + \boldsymbol{\partial}_{x'}\boldsymbol{\epsilon}(x')]}_{1+\,\mathrm{Tr}[\boldsymbol{\partial}_{x'}\boldsymbol{\epsilon}(x')]+\mathcal{O}[\epsilon^2(x')]}\ \big(g(x') + \epsilon^\gamma(x')\partial_{\gamma'}g(x') + \mathcal{O}[\epsilon^2(x')]\big)\\
&= g(x') + \big[\partial_{\mu'}\epsilon^\mu(x')\big]g(x') + \epsilon^\mu(x')\partial_{\mu'}g(x') + \mathcal{O}[\epsilon^2(x')]\\
&= g(x') + \partial_{\mu'}[\epsilon^\mu(x')g(x')] + \mathcal{O}[\epsilon^2(x')],
\end{aligned}
\tag{38}
$$

where to get to the second line we used $\boldsymbol{P}_{x\to x'} = \mathbb{1} + \boldsymbol{\partial}_{x'}\boldsymbol{\epsilon}(x')$, i.e. eq. (31) in matrix form. From eq. (38), we infer

$$
g'(x) = g(x) + \partial_\mu[\epsilon^\mu(x)g(x)] + \mathcal{O}[\epsilon^2(x)], \tag{39}
$$

from which in turn it follows that

$$
\delta g(x) = g'(x) - g(x) = \partial_\mu[\epsilon^\mu(x)g(x)] + \mathcal{O}[\epsilon^2(x)], \tag{40}
$$

or in unshortened notation,

$$
\delta\sqrt{-\det[\boldsymbol{g}(x)]} = \partial_\mu\Big[\epsilon^\mu(x)\sqrt{-\det[\boldsymbol{g}(x)]}\Big] + \mathcal{O}[\epsilon^2(x)]. \tag{41}
$$

3 Action Principle

a) Consider a field $\phi(x)$ and an action in d spacetime dimensions with Minkowskian signature $(-1,+1,...,+1)$ of the form

$$
S[\phi] = \int \mathrm{d}^d x\mathcal{L}\big(\phi(x), \partial_\mu\phi(x)\big). \tag{42}
$$

The simplest example is the action for the free scalar field

$$
S[\phi] = -\frac{1}{2}\int \mathrm{d}^d x\big(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\big). \tag{43}
$$

The variation of the general action (42) is defined as

$$
\delta S[\phi] = \int \mathrm{d}^d x\left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right). \tag{44}
$$

Using integration by parts and neglecting boundary terms show that the action principle $\delta S[\phi] = 0$ implies the Euler Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi}. \tag{45}$$

b) Use this result to derive the Klein-Gordon equation as the Euler-Lagrange equation of the free scalar field action (43).

a) Using partial integration on the second term in the variation of the action (44), $\delta S[\phi]$ becomes

$$\delta S[\phi] = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \bigg|_{\partial \mathbb{R}^d}}_{0} + \int \mathrm{d}^d x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi. \tag{46}$$

Since we assume we are working with localized *physical* systems, we take the boundary terms to vanish at spatial and temporal infinity, i.e. the term in front can be disregarded.

At this point, we resort to Hamilton's principle of a stationary action, $\delta S[\phi] = 0$, which we require to hold for all possible variations $\delta\phi$ of $\phi(x)$. $\delta S[\phi] = 0$ can hence only be true in general if the integrand itself vanishes. We thus arrive at the renowned Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0. \tag{47}$$

b) Comparing eqs. (42) and (43), we see that the Lagrangian $\mathcal{L}\big(\phi(x), \partial_\mu \phi(x)\big)$ of the free scalar field is given by

$$\mathcal{L}\big(\phi(x), \partial_\mu \phi(x)\big) = -\frac{1}{2}\big(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2\big). \tag{48}$$

Inserting this expression into eq. (47), we obtain as equation of motion,

$$-m^2 \phi + \partial_\mu \partial^\mu \phi = 0, \tag{49}$$

which written in a more familiar form reads

$$(\Box_x - m^2)\phi(x) = 0. \tag{50}$$