Group Theory – Problem Sheet 2

Exercise sheet 2, Group Theory, Prof. Daniel Roggenkamp

Solutions by Janosh Riebesell, 02.06.2015

Exercise 1 (Representations)

Recall the following notions from the lecture:

a) What is a representation?

A (linear) *representation* of a group G on a vector space V over a field K is a *group homomorphism* from G to $\mathrm{GL}(V)$, i.e. a map

$$\rho : G \longrightarrow \mathrm{GL}(V) = \mathrm{Aut}(V) = \{\text{all invertible, linear transf. } V \to V\}$$

such that $\rho(g_1)\rho(g_2) = \rho(g_1 g_2) \quad \forall g_1, g_2 \in G$.

V is called the representation space and $\dim(V)$ the dimension of the representation.

b) What is a homomorphism (intertwiner) between two representations?

A *group homomorphism* between two groups is a map that preserves the group operation, i.e. given two groups $(G, \cdot)$ and $(H, *)$, a group homomorphism φ with

$$\varphi : G \to H$$

fulfills $\varphi(g_1 \cdot g_2) = \varphi(g_1) * \varphi(g_2) \quad \forall g_1, g_2 \in G$.

Let V_1 and V_2 be the representation spaces of two (linear) representations of the group G,

$$\rho_1 : G \to \mathrm{GL}(V_1), \quad \rho_2 : G \to \mathrm{GL}(V_2).$$

Then, a linear map $f : V_1 \to V_2$ is called an *intertwiner* of the representations ρ_1 and ρ_2 if it commutes under the group action of G, i.e. if $f \circ \rho_1(g) = \rho_2(g) \circ f \quad \forall g \in G$.

An intertwiner's property can be visualized by the following diagram.

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ V_2 & \xrightarrow[\rho_2(g)]{} & V_2 \end{array}$$

Notably, an intertwiner must fulfill this diagram for every $g \in G$. If f is invertible (only possible if $\dim(V_1) = \dim(V_2)$), we have $\rho_1(g) = f^{-1} \circ \rho_2(g) \circ f$ and the representations ρ_1 and ρ_2 are said to be equivalent.

c) What is a subrepresentation?

A subspace $W \subset V$ of the representation space V of a representation ρ is called a *subrepresentation* if it is invariant under the group action of the represented group G, where invariant means that $gw \in W \quad \forall g \in G$ and $\forall w \in W$.

The restriction of ρ to W, $\rho(W)|_W$, is also referred to as subrepresentation, if $W \subset V$ is an invariant subspace: $\rho|_W : G \to \mathrm{GL}(W)$ is then a representation on its own.

d) What are reducible and irreducible representations?

A representation $\rho : G \to \mathrm{GL}(V)$ is said to be *irreducible* if it has only the trivial subrepresentations $\{0\}$ and V. If there is a non-trivial subrepresentation, ρ is reducible.

e) Recall Schur's lemma.

If ρ_1 and ρ_2 are two finite-dimensional irreducible representations of a group G and $f : V_1 \to V_2$ is an intertwiner between their representation spaces, then either $f \equiv 0$ or f is invertible, i.e. an isomorphism.

In the special case of $V_1 \cong V_2$, it further yields $f = \lambda \mathrm{id}_V$ (with $\lambda \in \mathbb{C}$, if V_1 and V_2 are vector spaces over $\mathbb{C}$).

f) Argue that complex representations of finite groups are fully decomposable.

A representation $\rho : G \to \mathrm{GL}(V)$ of a group G is called *fully decomposable* if it can be decomposed into a direct sum of irreducible representations, $\rho = \rho_1 \oplus \cdots \oplus \rho_n$, where $\rho_i,\ i \in \{1, ..., n\}$ each irreducible. The above claim follows from this argument:

Let $\rho : G \to \mathrm{GL}(V)$ be a reducible complex representation (if it were irreducible, it would be fully decomposable by definition). Averaging any Hermitian inner product over the finite group, $\langle v, w\rangle_G := \sum_{g\in G}\langle \rho(g)v, \rho(g)w\rangle$, gives an inner product with respect to which ρ is unitary, $\rho : G \to U(V)$.

$$\implies \exists W \subset V \text{ invariant subspace.}$$

We can define $W^\perp = \{v \in V \mid \langle v, w\rangle = 0 \quad \forall w \in W\}$. Then, since

$$\forall g \in G, \forall w \in W, \forall v \in W^\perp : \quad 0 = \langle v, w\rangle \overset{W \text{ invariant subspace}}{=} \langle v, \rho(g)w\rangle \overset{\rho \text{ unitary repr.}}{=} \langle \rho(g^{-1})v, w\rangle,$$

we know that $W^\perp$ is an invariant subspace as well. We infer that

$$\rho = \rho|_W \oplus \rho|_{W^\perp},$$

where $\rho|_W$ is still unitary, therefore we may split again. The above scheme can be applied repeatedly until the subrepresentations become irreducible. Finally

$$\rho = \bigoplus_i \rho_i, \quad \text{with } \rho_i \text{ irreducible } \forall i.$$

One may say that irreducible representations are the building blocks of unitary representations.

Exercise 2 (Representations of abelian groups)

a) Show that all irreducible representations of a finite group G are one-dimensional if and only if G is abelian. (*Hint:* What is the sum of the squares of the dimensions of the irreducible representations?)

To be shown:

$$\dim(\rho_i) = 1 \quad \forall \rho_i : G \to \mathrm{GL}(V_i) \text{ irreducible}, i \in \{1, ..., n\} \quad \iff \quad G \text{ abelian.}$$

"$\implies$": Assume all irreducible representations have $d_i = \dim(\rho_i) = 1$. Then we have

$$|G| \overset{!}{=} \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} 1 = n, \quad \text{with } n : \text{number of irred. repr.}$$

We also have that the number of irreducible representations equals the number of conjugacy classes of G, i.e.

$$n \overset{!}{=} \text{number of conjugacy classes of } G.$$

Therefore, 'number of conjugacy classes' $= n = |G|$, i.e. every element of G forms its own conjugacy class.

$$\implies hgh^{-1} = g \quad \forall g, h \in G \quad \implies \quad hg = gh \quad \forall g, h \in G \quad \iff \quad G \text{ is abelian.}$$

"$\Longleftarrow$": Assume G is abelian $\iff gh = hg \quad \forall g, h \in G$. Then, since every representation is a homomorphism, we have

$$\rho(g_1)\rho(g_2) \overset{\rho \text{ homomorphism}}{=} \rho(g_1 g_2) \overset{G \text{ abelian}}{=} \rho(g_2 g_1) = \rho(g_2)\rho(g_1).$$

Applying the special case of Schur's lemma, which states that if $f : V \to V$ commutes with $\rho(g) \quad \forall g \in G$, i.e.

$$f \circ \rho(g) = \rho(g) \circ f \quad \forall g \in G \quad \implies \quad f = \lambda \mathrm{id}_V \quad \text{for some } \lambda \in \mathbb{C},$$

we obtain

$$\rho(g) = \lambda_g \mathrm{id}_V = \begin{pmatrix} \lambda_g & & 0 \\ & \ddots & \\ 0 & & \lambda_g \end{pmatrix} = \bigoplus_i \rho_i, \quad \text{with } \rho_i \text{ irreducible repr. of} \dim(\rho_i) = 1.$$

b) What are the irreducible representations of the cyclic group $\mathbb{Z}_n$?

Since $\mathbb{Z}_n$ is abelian, we know from part a) that every irreducible subrepresentation of $\mathbb{Z}_n$ is one-dimensional and that there are exactly $|\mathbb{Z}_n| = n$ of them.

Let $\mathrm{id}_{\mathbb{C}}^{1/n}$ denote the n-th root of unity in $\mathbb{C}$, then the map

$$\rho : \mathbb{Z}_n \longrightarrow \mathbb{C}^*, \quad \rho(g^j) = \mathrm{id}_{\mathbb{C}}^{j/n} \quad \forall g^j \in \mathbb{Z}_n$$

is a group homomorphism. Since $\mathrm{id}_{\mathbb{C}}^{k/n}$, $k \in \{1, ..., n\}$ are the first through n-th roots of unity in $\mathbb{C}$, the n irreducible subrepresentations of $\mathbb{Z}_n$ with dimension one are

$$\rho_k : G \longrightarrow \mathbb{C}^*, \quad \rho_k(g^j) = \mathrm{id}_{\mathbb{C}}^{jk/n} = e^{2\pi i jk/n}, \quad \text{with } k \in \{1, ..., n\}.$$

Exercise 3 (Orthogonality relation)

Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible (finite-dimensional complex) representations of a finite group G. Choose bases of V_1 and V_2 and let $(\rho_1(g))_{ij}$ and $(\rho_2(g))_{ab}$ be the representation matrices with respect to these bases. Then use Schur's lemma to show:

a) If ρ_1 and ρ_2 are inequivalent, then

$$\sum_{g \in G} (\rho_1(g^{-1}))_{ij} (\rho_2(g))_{ab} = 0$$

for all i, j, a, b.

To show this, we consider the matrix

$$M_X = \sum_{g\in G} \underbrace{\rho_1(g)^{-1}}_{\rho_1(g^{-1})} X\rho_2(g),$$

where X is an arbitrary matrix with d_1 rows and d_2 columns, d_1 and d_2 being the dimensionalities of the representations ρ_1 and ρ_2, s.t. M_X is a $d_1 \times d_2$-matrix. Now multiplying M_X from the left and right with arbitrary matrices in the representations ρ_1 and ρ_2 respectively,

$$\begin{aligned}
\rho_1^{-1}(h)M_X\rho_2(h) &= \sum_{g\in G} \underbrace{\rho_1^{-1}(h)}_{\rho_1(h^{-1})}\underbrace{\rho_1^{-1}(g)}_{\rho_1(g^{-1})} X\rho_2(g)\rho_2(h) \quad \text{for some } h \in G \\
&= \sum_{g\in G} \rho_1(\underbrace{h^{-1}g^{-1}}_{(gh)^{-1}})X\rho_2(gh) \qquad\qquad g \mapsto gh^{-1} \\
&= \sum_{g\in G} \rho_1(g^{-1})X\rho_2(g) = M_X,
\end{aligned}$$

we find that M_X is an intertwiner between the representations ρ_1 and ρ_2, i.e. $\rho_1(g)M_X = M_X\rho_2(g)$. Since ρ_1 and ρ_2 are taken to be inequivalent, i.e.

$$\nexists M_X : \quad \rho_1(g) = M_X\rho_2(g)M_X^{-1} \quad (M_X \text{ not invertible}),$$

we know by Schur's lemma that $M_X = 0$. Therefore, writing element-wise, we have

$$0 = (M_X)_{ib} = \sum_{g\in G} \left(\rho_1(g^{-1})\right)_{ik} X_{kl} \left(\rho_2(g)\right)_{lb}.$$

Since the matrix X was arbitrary, we may choose any form. We pick one X for every tuple of j, a given by

$$(X_{kl})_{ja} = \delta_{kj}\delta_{la}$$

to conclude that

$$\sum_{g\in G} \left(\rho_1(g^{-1})\right)_{ik} \delta_{kj}\delta_{la} \left(\rho_2(g)\right)_{lb} = \sum_{g\in G} \left(\rho_1(g^{-1})\right)_{ij} \left(\rho_2(g)\right)_{ab} \overset{!}{=} 0$$

for all i, j, a, b.

b) If on the other hand $\rho_1 = \rho_2$, then

$$\frac{1}{|G|} \sum_{g\in G} \left(\rho(g^{-1})\right)_{ij} (\rho(g))_{ab} = \frac{1}{\dim(V)} \delta_{ib}\delta_{ja}.$$

In the case of $\rho_1 = \rho_2 =: \rho$, ρ is still an irreducible representation of G and M_X with

$$M_X = \sum_{g\in G} \rho(g)^{-1} X \rho(g)$$

is still an intertwiner, so that we may apply Schur's lemma for the special case of $V_1 = V_2 =: V$ to conclude

$$M_X = \lambda \mathrm{id}_V \quad \text{for some } \lambda \in \mathbb{C} \text{ for every } g \in G.$$

Taking the trace of M_X, we obtain

$$\mathrm{Tr}(M_X) = \mathrm{Tr}_V\left(\sum_{g\in G} \rho(g^{-1})X\rho(g)\right) \overset{\text{linearity of trace}}{=} \sum_{g\in G} \mathrm{Tr}_V(\rho(g^{-1})X\rho(g))$$

$$\overset{\text{cyclicity of trace}}{=} \sum_{g\in G} \mathrm{Tr}_V\left(\underbrace{\rho(g)\rho(g^{-1})}_{\rho(gg^{-1})=\rho(e)=\mathrm{id}_V} X\right) = \sum_{g\in G} \mathrm{Tr}_V\left(\underbrace{\mathrm{id}_V X}_{X}\right) = |G|\ \mathrm{Tr}_V(X),$$

$$\mathrm{Tr}(M_X) = \mathrm{Tr}(\lambda \mathrm{id}_V) = \lambda\ \mathrm{Tr}(\mathrm{id}_V) = \lambda \dim(V) \quad\Longrightarrow\quad \lambda = \frac{|G|}{\dim(V)} \mathrm{Tr}_V(X).$$

Choosing again $\left(X_{ja}\right)_{kl} = \delta_{kj}\delta_{al}$, we obtain

$$\begin{aligned}(M_X)_{ib} &= \sum_{g\in G} \left(\rho(g^{-1})\right)_{ik} \delta_{kj}\delta_{al} (\rho(g))_{lb} = \sum_{g\in G} \left(\rho(g^{-1})\right)_{ij} (\rho(g))_{ab} \\ &= \lambda\delta_{ib} = \frac{|G|}{\dim(V)} \mathrm{Tr}_V(X)\delta_{ib} = \frac{|G|}{\dim(V)} \underbrace{\mathrm{Tr}_V\left(\delta_{kj}\delta_{al}\right)}_{\delta_{kj}\delta_{ak}=\delta_{ja}} \delta_{ib} = \frac{|G|}{\dim(V)} \delta_{ib}\delta_{ja} \\ \Longrightarrow \frac{1}{|G|} \sum_{g\in G} &\left(\rho(g^{-1})\right)_{ij} (\rho(g))_{ab} = \frac{1}{\dim(V)} \delta_{ib}\delta_{ja}.\end{aligned}$$

c) Deduce that

$$\frac{1}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1}h)\chi_{\rho_2}(g) = \begin{cases} \dfrac{\chi_{\rho_1}(h)}{\dim(\rho_1)} & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{otherwise.} \end{cases}$$

We express the characters $\chi_{\rho_1}(g^{-1}h) \coloneqq \mathrm{Tr}_{V_1}(\rho_1(g^{-1}h))$ and $\chi_{\rho_2}(g) \coloneqq \mathrm{Tr}_{V_2}(\rho_2(g))$ as

$$\chi_{\rho_1}(g^{-1}h) = \sum_{i,j} \left(\rho_1(g^{-1})\right)_{ij} (\rho_1(h))_{ji} \quad \text{and} \quad \chi_{\rho_2}(g) = \sum_k \left(\rho_2(g)\right)_{kk}$$

to obtain

$$\begin{aligned}\frac{1}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1}h)\chi_{\rho_2}(g) &= \frac{1}{|G|} \overbrace{\sum_{g\in G} \sum_{i,j,k}}^{\longleftarrow} \left(\rho_1(g^{-1})\right)_{ij} \overbrace{(\rho_1(h))_{ji} (\rho_2(g))_{kk}}^{\longrightarrow} \\ = \frac{1}{|G|} \sum_{i,j,k} \sum_{g\in G} &\left(\rho_1(g^{-1})\right)_{ij} (\rho_2(g))_{kk} (\rho_1(h))_{ji} \\ = \begin{cases} \dfrac{1}{\dim(V)} \displaystyle\sum_{i,j,k} \delta_{ik}\delta_{jk} (\rho(h))_{ji} = \frac{1}{\dim(V)} \sum_k (\rho(h))_{kk} = \frac{\chi_\rho(h)}{\dim(V)} & \text{if } \rho_1 = \rho_2 =: \rho \text{ using b)} \\ \dfrac{1}{|G|} \displaystyle\sum_{i,j,k} 0 \cdot (\rho(h))_{ji} = 0 & \text{otherwise using a).} \end{cases}\end{aligned}$$

Exercise 4 (Symmetric and antisymmetric products)

Let (ρ, V) be a finite-dimensional representation. Consider the tensor product representation $\rho \otimes \rho$ on $V \otimes V$. Define $\sigma : V \otimes V \to V \otimes V$ by $\sigma(v_1 \otimes v_2) = v_2 \otimes v_1 \quad \forall v_1, v_2 \in V$.

a) Show that the tensor product decomposes into the sum $V \otimes V = S^2V \oplus \Lambda^2 V$ of spaces of symmetric and antisymmetric tensors $S^2V = \{v \in V \otimes V \mid \sigma(v) = v\}$ and $\Lambda^2 V = \{v \in V \otimes V \mid \sigma(v) = -v\}$, and that these are invariant subspaces in $V \otimes V$. (*Hint:* Consider the projectors $\frac{1}{2}(1 \pm \sigma) : V \otimes V \to V \otimes V$.)

Following the hint, we examine $\pi_\pm \coloneqq \frac{1}{2}(\mathrm{id}_{V\otimes V} \pm \sigma) : V \otimes V \to V \otimes V$,

$$\pi_\pm^2 = \frac{1}{4}\left(\mathrm{id}_{V\otimes V} \pm \sigma\right)^2 = \frac{1}{4}\left(\underbrace{\mathrm{id}^2_{V\otimes V}}_{\mathrm{id}_{V\otimes V}} \pm 2\underbrace{\mathrm{id}_{V\otimes V}\sigma}_{\sigma} + \underbrace{\sigma^2}_{\mathrm{id}_{V\otimes V}}\right) = \frac{1}{4}\left(2\mathrm{id}_{V\otimes V} \pm 2\sigma\right) = \pi_\pm.$$

As hinted, we find $\pi_\pm$ to be projectors. In addition,

$$\pi_+ + \pi_- = \frac{1}{2}\left(\mathrm{id}_{V\otimes V} + \sigma\right) + \frac{1}{2}\left(\mathrm{id}_{V\otimes V} - \sigma\right) = \mathrm{id}_{V\otimes V}.$$

Therefore, the tensor product $V \otimes V$ can be decomposed as

$$V \otimes V = \mathrm{Im}(\pi_+) \oplus \mathrm{Im}(\pi_-).$$

It remains to be shown that $\mathrm{Im}(\pi_+) = S^2V$ and $\mathrm{Im}(\pi_-) = \Lambda^2 V$.

$$\begin{aligned}
S^2V \subset \mathrm{Im}(\pi_+): \quad & \forall v \in S^2V : \sigma(v) = v \\
& \Longrightarrow \forall v \in S^2V : \pi_+(v) = \frac{1}{2}\left(\mathrm{id}_{V\otimes V}(v) + \sigma(v)\right) = \frac{1}{2}\left(v + v\right) = v \\
& \Longrightarrow S^2V \subset \mathrm{Im}(\pi_+), \text{since } \pi_+ \text{ is projector,} \\
\mathrm{Im}(\pi_+) \subset S^2V: \quad & \forall v \in \mathrm{Im}(\pi_+) : \pi_+(v) = v, \text{since } \pi_+ \text{ is projector} \\
& \Longrightarrow \pi_+(v) = \frac{1}{2}\left(\mathrm{id}_{V\otimes V}(v) + \sigma(v)\right) = \frac{1}{2}\left(v + \sigma(v)\right) \overset{!}{=} v \Longleftrightarrow \sigma(v) = v \\
& \Longrightarrow \mathrm{Im}(\pi_+) \subset S^2V \\
\Longrightarrow \mathrm{Im}(\pi_+) = S^2V.
\end{aligned}$$

This goes through analogously for $\mathrm{Im}(\pi_-) = \Lambda^2 V$.

$$\Longrightarrow V \otimes V = S^2V \oplus \Lambda^2 V.$$

To further show that S^2V and $\Lambda^2 V$ are invariant subspaces in $V \otimes V$, we remember a remark from the lecture, which states that image and kernel of every intertwiner form invariant subspaces. Since σ is an intertwiner because it fulfills $\rho \otimes \rho(g) \circ \sigma = \sigma \circ \rho \otimes \rho(g)$, so is $\pi_\pm$ w.r.t. $\rho \otimes \rho$. Therefore, S^2V and $\Lambda^2 V$ are invariant subspaces.

b) Show that the characters of the corresponding subrepresentations can be expressed as $\chi_{S^2V}(g) = \frac{1}{2}\left[\left(\chi_V(g)\right)^2 + \chi_V(g^2)\right]$, $\chi_{\Lambda^2V}(g) = \frac{1}{2}\left[\left(\chi_V(g)\right)^2 - \chi_V(g^2)\right]$. (A character is defined as the trace over a representation, $\chi_V(g) = \mathrm{tr}_V(\rho(g))$.)

A subrepresentation is formed by restricting a representation ρ to a subspace W of its representation space V, $\rho|_W$. Since $\pi_\pm$ are projectors onto their respective subspaces, we may write the subrepresentations $\rho \otimes \rho|_{S^2V}$ and $\rho \otimes \rho|_{\Lambda^2V}$ as

$$\rho \otimes \rho|_{S^2V} = \rho \otimes \rho\, \pi_+, \quad \rho \otimes \rho|_{\Lambda^2V} = \rho \otimes \rho\, \pi_-.$$

Making use of $\pi_\pm$ also being intertwiners, we may express the characters by

$$\begin{aligned}
\chi_{S^2V}(g) &= \mathrm{tr}_{V\otimes V}(\pi_+\, \rho \otimes \rho(g)) = \frac{1}{2}\left[\mathrm{tr}_{V\otimes V}(\mathrm{id}_{V\otimes V}\, \rho \otimes \rho(g)) + \mathrm{tr}_{V\otimes V}(\sigma\, \rho \otimes \rho(g))\right], \\
\chi_{\Lambda^2V}(g) &= \mathrm{tr}_{V\otimes V}(\pi_-\, \rho \otimes \rho(g)) = \frac{1}{2}\left[\mathrm{tr}_{V\otimes V}(\mathrm{id}_{V\otimes V}\, \rho \otimes \rho(g)) - \mathrm{tr}_{V\otimes V}(\sigma\, \rho \otimes \rho(g))\right].
\end{aligned}$$

We can simplify both summands:

$$\mathrm{tr}_{V\otimes V}(\mathrm{id}_{V\otimes V}\, \rho \otimes \rho(g)) = \mathrm{tr}_V(\rho(g))\mathrm{tr}_V(\rho(g)) = \left[\mathrm{tr}_V(\rho(g))\right]^2 = \chi_V^2(g).$$

To calculate the second, we need to choose a basis, say e_i, for V. Then

$$\begin{aligned}\mathrm{tr}_{V\otimes V}(\sigma\,\rho\otimes\rho(g)) &= \mathrm{tr}_{V\otimes V}\big(\sigma\,\rho(g)\otimes\rho(g)(e_i\otimes e_j)\big)\\ &= \mathrm{tr}_{V\otimes V}\big(\rho(g)\otimes\rho(g)(e_j\otimes e_i)\big) = \mathrm{tr}_{V\otimes V}\big(\rho_{aj}(g)\rho_{bi}(g)e^b\otimes e^a\big)\\ &= \mathrm{tr}_V(\rho(g)\cdot\rho(g)) = \mathrm{tr}_V(\rho(g^2)).\end{aligned}$$

Reinserting the simplifications into $\chi_{S^2V}(g)$ and $\chi_{\Lambda^2V}(g)$, we arrive at the desired result:

$$\chi_{S^2V}(g) = \frac{1}{2}\left[\chi_V^2(g) + \chi_V(g^2)\right],$$
$$\chi_{\Lambda^2V}(g) = \frac{1}{2}\left[\chi_V^2(g) - \chi_V(g^2)\right].$$

Exercise 5 (Representations of D_5)

Consider the dihedral group

$$D_5 = \{e = R_0, R_1, R_2, R_3, R_4, S_0, S_1, S_2, S_3, S_4\}$$

of symmetries of the regular pentagon in the plane. Its character table is given by

D_5	$C_0 = \{e\}$	$C_1 = \{R_1, R_4\}$	$C_2 = \{R_2, R_3\}$	$C_3 = \{S_0, ..., S_4\}$
χ_0	1	1	1	1
χ_1	1	1	1	-1
χ_2	2	α	β	0
χ_3	2	β	α	0

a) Use orthogonality of characters to show that $\alpha = \frac{1}{2}\left(-1+\sqrt{5}\right)$ and $\beta = \frac{1}{2}\left(-1-\sqrt{5}\right)$.

The orthogonality relation of two characters is given by

$$\sum_a \overline{\chi_i(C_a)}\chi_j(C_a)\frac{|C_a|}{|G|} = \delta_{ij}, \quad \text{where } \overline{\chi(g)} = \chi(g^{-1}).$$

The orthogonality of rows works just as well here, but we apply the orthogonality relation for columns given by

$$\sum_i \overline{\chi_i(C_a)}\chi_i(C_b) = \frac{|G|}{|C_a|}\delta_{ab}.$$

Multiplying the first and second (or third), and the second and third columns, we obtain the constraints

$$\begin{aligned}
1\cdot 1 + 1\cdot 1 + 2\alpha + 2\beta = 2 + 2\alpha + 2\beta = 0 \quad &\Longleftrightarrow \quad 1 + \alpha + \beta = 0,\\
1\cdot 1 + 1\cdot 1 + \alpha\beta + \beta\alpha = 2 + 2\alpha\beta = 0 \quad &\Longleftrightarrow \quad 1 + \alpha\beta = 0,\\
\Longrightarrow \beta = -(1+\alpha), \quad &\qquad\quad 1 + \alpha\beta = 1 - \alpha(1+\alpha) = 1 - \alpha - \alpha^2 = 0\\
\Longleftrightarrow \left(\alpha + \frac{1}{2}\right)^2 = \frac{5}{4} \quad &\Longrightarrow \quad \alpha = -\frac{1}{2} \pm \sqrt{\frac{5}{4}} = \frac{1}{2}\left(-1 \pm \sqrt{5}\right)\\
&\Longrightarrow \quad \beta = -\left(1 + \frac{1}{2}\left(-1 \pm \sqrt{5}\right)\right) = \frac{1}{2}\left(-1 \mp \sqrt{5}\right).
\end{aligned}$$

b) Consider the following representations: $\rho_2 \otimes \rho_2$, $\Lambda^2\rho_2$, $S^2\rho_2$. Determine their characters and decompose them into irreducible representations.

Characters are multiplicative with respect to tensor product, i.e.

$$\chi_{\rho_2\otimes\rho_2}(g) = \chi_{\rho_2}(g) \cdot \chi_{\rho_2}(g).$$

Applying our result from exercise 4.b), i.e.

$$\chi_{S^2V}(g) = \frac{1}{2}\left(\chi_V^2(g) + \chi_V(g^2)\right), \quad \chi_{\Lambda^2V}(g) = \frac{1}{2}\left(\chi_V^2(g) - \chi_V(g^2)\right),$$

we may write a new character table for $\chi_{\rho_2\otimes\rho_2}$, $\chi_{S^2\rho_2}$, $\chi_{\Lambda^2\rho_2}$:

D_5	C_0	C_1	C_2	C_3
$\chi_{\rho_2\otimes\rho_2}$	4	$\alpha^2 = 1-\alpha$	$\beta^2 = 1-\beta$	0
$\chi_{S^2\rho_2}$	3	$-\alpha$	$-\beta$	1
$\chi_{\Lambda^2\rho_2}$	1	1	1	-1

Using $\alpha + \beta = -1$, i.e. $1 - \alpha = 2 + \beta$ and $1 - \beta = 2 + \alpha$, we find $\chi_{\rho_2\otimes\rho_2} = \chi_0 + \chi_1 + \chi_3$. Therefore, $\rho_2 \otimes \rho_2 \cong \rho_0 \oplus \rho_1 \oplus \rho_3 = S^2\rho_2 \oplus \Lambda^2\rho_2$, where

$$\begin{aligned} \chi_{S^2\rho_2} = \chi_0 + \chi_3 \quad &\Longrightarrow \quad S^2\rho_2 \cong \rho_0 \oplus \rho_3, \\ \chi_{\Lambda^2\rho_2} = \chi_1 \quad &\Longrightarrow \quad \Lambda^2\rho_2 \cong \rho_1. \end{aligned}$$