Group Theory – Problem Sheet 3

Exercise sheet 3, Group Theory, Prof. Daniel Roggenkamp

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Exercise 1 (Representations of S_n)

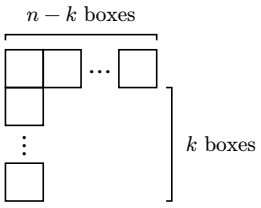
a) Use the hook rule to determine the dimension of the irreducible representation associated to the partition

$$\lambda = (n-k, \underbrace{1, ..., 1}_{k \text{ times}}, 0, ..., 0).$$

The hook length of a box in a Young tableau is the number of boxes beneath or to the right of the box in the diagram, counting the box itself.

Let ρ^λ be the irreducible repr. of S_n associated with the partition λ, then the hook rule states the dimension of ρ^λ to be $n!$ over the hook lengths in the Young diagram of shape λ, i.e. in this instance

$$\dim(\rho^\lambda) = \frac{n!}{n \cdot (n-k-1)! \cdot k!} = \binom{n-1}{k}, \quad \text{Young diagram:}$$



since the corner box has hook length n, the remaining boxes of the first row have hook lengths $n-k-1, ..., 1$ and the remaining boxes of the first column $k, ..., 1$.

b) Construct the character table of S_4 by using the Frobenius formula.

According to the Frobenius formula, the characters of the irreducible repr. of S_n associated with the partition $\lambda = (\lambda_1, ..., \lambda_k)$ of n are given by

$$\chi_\lambda(C_i) = \left[\Delta(x) \prod_{j=1} P_j(x)^{i_j} \right]_{(l_1,...,l_k)},$$

where P_j is the power sum $P_j(x) = x_1^j + \cdots + x_k^j$, k being the number of rows in the Young tableau of λ, and $\Delta(x)$ is the discriminant

$$\Delta(x) = \prod_{1 \le i < j \le k} (x_i - x_j) = \det \begin{pmatrix} 1 & x_k & \cdots & x_k^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1 & \cdots & x_1^{k-1} \end{pmatrix}$$

of the set of k independent variables $\{x_1, ..., x_k\}$. Furthermore, $i = (i_1, ..., i_n)$ is an n-tuple of non-negative integers s.t. $\sum_r r i_r = n$ so that C_i may denote the conjugacy class consisting of elements with cycle type $(i_1, ..., i_n)$, and $[f(x)]_{(l_1,...,l_k)}$ is the coefficient of $x_1^{l_1} \cdots x_k^{l_k}$ in the power series $f(x) = f(x_1, ..., x_k)$, where $l = (l_1, ..., l_k)$ is a k-tuple of non-negative integers.

Applying the Frobenius formula yields the following character table:

S_4	$C_1 = [(1)]$	$C_2 = [(12)]$	$C_3 = [(123)]$	$C_4 = [(1234)]$	$C_5 = [(12)(34)]$
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ	3	1	0	-1	-1
$\chi_{\text{sgn}} \otimes \chi$	3	-1	0	1	-1
χ_{other}	2	0	-1	0	2

Exercise 2 (Lie groups and Lie algebras)

f) How are the notions of homomorphisms of Lie groups and Lie algebras related?

Let $\phi : G \to H$ be a Lie group homomorphism. With ϕ_* we denote its derivative at the identity of G. If we pick $\mathfrak{g}$, $\mathfrak{h}$, the Lie algebras of G and H, as the tangent spaces at their respective identities, then ϕ_* is a map between these two Lie algebras,

$$\phi_* : \mathfrak{g} \to \mathfrak{h},$$

which fulfills all the requirements of a Lie algebra homomorphism, i.e. it is a linear map that preserves the Lie bracket.

a) What is a Lie group?

A Lie group is a group G which is also a differentiable (also called smooth) manifold s.t. the group operations are compatible with the smooth group structure, i.e. s.t.

$$\left.\begin{array}{l} m : G \times G \to G \ \text{(multiplication)} \\ (\cdot)^{-1} : G \to G \quad \text{(inversion)} \end{array}\right\} \quad \text{are smooth maps.}$$

b) What is a Lie algebra?

A Lie algebra is a vector space V with a non-associative, bilinear, skew-symmetric map $[\cdot, \cdot] : V \times V \to V$ (called the Lie bracket) which satisfies the Jacobi identity.

c) What are left-invariant vector fields? Describe them for matrix groups.

A vector field X_v on a Lie group G is left-invariant if

$$\big(\mathrm{d}L_g\big)(X_v(h)) = X_v\big(L_g(h)\big) = X_v(gh) \quad \forall g, h \in G,$$

where $\mathrm{d}L_g$ is the derivative of the smooth map L_g, and L_g denotes the group action of the Lie group G defined by left multiplication, i.e. $L_g : G \to G$, $L_g(h) = gh$ $\forall g, h \in G$ ($L_e = \mathrm{id}_G$).

d) How does one obtain the structure of a Lie algebra on a tangent space of a Lie group at the identity?

By finding the vector space of all left-invariant vector fields.

Reasoning: Since the vector space of all left-invariant vector fields is isomorphic (as a vector space) to T_eG, the tangent space of a Lie group G at the identity e, and it in turn is equipped with the structure of a Lie algebra, defined by

$$[v, w] = X_{[v,w]} = [X_v, X_w] \quad \forall v, w \in T_eG,$$

this yields the desired information.

e) Show that the Lie bracket on the tangent space of the identity of matrix groups satisfies the Jacobi identity.

Every matrix (Lie) group is a subgroup of the general linear group $\mathrm{GL}(n,\mathbb{K})$,

$$\mathrm{GL}(n,\mathbb{K}) = \{A \in \mathrm{Mat}(n,n,\mathbb{K}) \mid \det(A) \neq 0\},$$

i.e. the group of all invertible $n \times n$ matrices over $\mathbb{K}$. Since the (standard) Lie bracket for Lie algebras of the matrix groups is simply the matrix commutator given by $[A,B] = AB - BA$ for $A, B \in \mathrm{Mat}(n,n,\mathbb{K})$, we calculate (cancelling terms marked in matching colors)

$$\begin{aligned}[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = {} & ABC - BCA - ACB + CBA \\ & + BCA - CAB - BAC + ACB \\ & + CAB - ABC - CBA + BAC = 0,\end{aligned}$$

which holds for all $A, B, C \in \mathrm{Mat}(n,n,\mathbb{K})$ and therefore for all possible elements of matrix groups.

Exercise 3 (SU(2))

Show that $\mathrm{SU}(2) = \left\{A \in \mathrm{Mat}(2,2,\mathbb{C}) \mid A^\dagger A = \mathbb{1}_2, \det A = 1\right\}$ can be parametrized in the following way

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

(Hence, SU(2) is homeomorphic to the three sphere S^3, and in particular simply connected.)

We write an element $A \in \mathrm{SU}(2)$ as $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}$. Then

$$A^\dagger A = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & \overline{a}c + \overline{b}d \\ a\overline{c} + b\overline{d} & |c|^2 + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The above property of A as an element of $\mathrm{SU}(2)$ yields the following three constraints on $a, b, c, d \in \mathbb{C}$:

$$\text{i) } |a|^2 + |b|^2 = 1, \quad \text{ii) } |c|^2 + |d|^2 = 1, \quad \text{iii) } a\overline{c} + b\overline{d} = 0.$$

The condition $\det(A) = 1$ further supplies

$$\text{iv) } ad - bc = 1.$$

From iii), $a = -\frac{b\overline{d}}{\overline{c}}$ follows. Inserting into i), we obtain

$$\frac{|b|^2\ |d|^2}{|c|^2} + |b|^2 = 1 \quad \Longrightarrow \quad \frac{|c|^2}{|b|^2} = |c|^2 + |d|^2 = 1,$$

i.e. $|b| = |c|$. Similarly, we find $|a| = |d|$ by subtracting i) and ii):

$$|a|^2 + \underbrace{|b|^2 - |c|^2}_{0} - |d|^2 = |a|^2 - |d|^2 = 0.$$

Remark: We assumed above that $b \neq 0$. If $b = 0$, it becomes very simple. By iv), we know $ad = 1$ and by iii), $a\overline{c} = 0$. The latter implies $c = 0$, since $a \neq 0$, and A becomes diagonal with $a = \overline{d}$ because $ad = 1$.

Now, returning to the case $b \neq 0$, we discover that our findings are captured by the ansatz

$$a = e^{i\alpha}\cos\theta, \quad b = e^{i\beta}\sin\theta, \quad c = -e^{i\gamma}\sin\theta, \quad d = e^{i\delta}\cos\theta,$$

where $\alpha, \beta, \gamma, \delta, \theta \in \mathbb{R}$. Applying iv) to this ansatz,

$$ad - bc = e^{i(\alpha+\delta)}\cos^2\theta + e^{i(\beta+\gamma)}\sin^2\theta = 1 \quad \Longleftrightarrow \quad \alpha = -\delta, \quad \beta = -\gamma,$$

we finally arrive at

$$a = e^{i\alpha}\cos\theta = e^{-i\delta}\cos\theta = \overline{d}, \quad b = e^{i\beta}\sin\theta = e^{-i\gamma}\sin\theta = -\overline{c}.$$

Thus, SU(2) may be parametrized by

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

Exercise 4 (SU(2) versus SO(3))

a) Determine the Lie algebras of SU(2) and SO(3) and show that they are isomorphic.

The SU(2) Lie algebra is

$$\begin{aligned} \mathfrak{su}(2) &= \left\{ A \in \mathrm{Mat}(2,2,\mathbb{C}) \mid \mathrm{tr}(A) = 0, A^\dagger = -A \right\} \\ &= \left\{ \begin{pmatrix} ia & -\overline{z} \\ z & -ia \end{pmatrix} \mid a \in \mathbb{R}, z \in \mathbb{C} \right\}, \end{aligned}$$

which amounts to the vector space of all traceless, antihermitian 2×2 matrices over $\mathbb{C}$. It is $n^2 - 1 = 3$-dimensional and generated by the following matrices

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The Lie algebra of SO(3),

$$\mathfrak{so}(3) = \left\{ A \in \mathrm{Mat}(3,3,\mathbb{R}) \mid A^T = -A \right\} = \left\{ \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \mid \omega_x, \omega_y, \omega_z \in \mathbb{R} \right\},$$

is the vector space of all antisymmetric 3×3 matrices over $\mathbb{R}$. $\mathfrak{so}(3)$ is $\frac{n}{2}(n-1) = 3$-dimensional with a generating basis formed by the following three matrices

$$o_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad o_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad o_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Both assertions concerning the structure of the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ can be seen by selecting and differentiating smooth curves from their corresponding Lie groups:

- Let $U(t) \in \mathrm{SU}(2)$ be a smooth curve with $U(0) = \mathbb{1}_2$. Then $U(t)$ satisfies

  $$U^\dagger(t)U(t) = \mathbb{1}_2, \quad \det(U(t)) = 1.$$

  Differentiation yields

$$\frac{\mathrm{d}U^\dagger(t)}{\mathrm{d}t}\,U(t) + U^\dagger(t)\,\frac{\mathrm{d}U(t)}{\mathrm{d}t} = 0, \quad \mathrm{tr}\left(U^{-1}(t)\,\frac{\mathrm{d}U(t)}{\mathrm{d}t}\right) = 0,$$

which, for $t = 0$ and writing $\frac{\mathrm{d}U(t)}{\mathrm{d}t}\Big|_{t=0}$ as $u \in \mathfrak{su}(2)$, demonstrates the above claims:

$$u^\dagger + u = 0, \quad \mathrm{tr}(u) = 0.$$

- Let $O(t) \in \mathrm{SO}(3)$ be a smooth curve with $O(0) = \mathbb{1}_3$. Then $O(t)$ satisfies the defining relation of O(3)-elements $O^T(t)O(t) = \mathbb{1}_3$. Thus

$$\frac{\mathrm{d}O^T(t)}{\mathrm{d}t}\,O(t) + O^T(t)\,\frac{\mathrm{d}O(t)}{\mathrm{d}t} = 0$$

and so for $t = 0$ and $o := \frac{\mathrm{d}O(t)}{\mathrm{d}t}\Big|_{t=0}$: $o^T + o = 0$.

Now on to the isomorphy between $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$. We can show this relation using the group homomorphism $\Phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ to construct an isomorphism $\varphi : \mathfrak{su}(2) \to \mathfrak{so}(3)$.

We start by expressing Φ using abstract index notation. Let $f(\vec{x})$ be a function of position $\vec{x}$, $\underline{U} \in \mathrm{SU}(2)$ and $\underline{R} = \Phi(\underline{U}) \in \mathrm{SO}(3)$ the corresponding rotation matrix. Then

$$\begin{aligned} f(\underline{R}\vec{x}) &= f(\Phi(\underline{U})\vec{x}) = \underline{U} f(\vec{x}) \underline{U}^\dagger, \\ \underline{R}^i_j x^j \underline{\sigma}_i &= \underline{U} x^j \underline{\sigma}_j \underline{U}^\dagger. \end{aligned}$$

Differentiating at $t = 0$ and setting $\frac{\mathrm{d}\underline{R}}{\mathrm{d}t}\Big|_{t=0} = \underline{X} \in \mathfrak{so}(3)$, $\underline{U}(0) = \mathbb{1}_2$ and $\frac{\mathrm{d}\underline{U}}{\mathrm{d}t}\Big|_{t=0} = \underline{A} = a^i \underline{s}_i \in \mathfrak{su}(2)$ with $\underline{s}_i = -\frac{i}{2}\underline{\sigma}_i$, we obtain

$$\begin{aligned} \underline{X}^i_j x^j \underline{\sigma}_i &= \underline{A} x^j \underline{\sigma}_j + x^j \underline{\sigma}_j \underline{A}^\dagger, \\ x^j \underline{X}^i_j \underline{\sigma}_i &= x^j \big(\underline{A}\underline{\sigma}_j - \underline{\sigma}_j \underline{A}\big) = x^j \big[\underline{A}, \underline{\sigma}_j\big] = x^j \big[a^i \underline{s}_i, \underline{\sigma}_j\big] \\ &= -\frac{i}{2}\, x^j a^i \big[\underline{\sigma}_i, \underline{\sigma}_j\big] = -\frac{i}{2}\, x^j a^i \Big(2i\varepsilon^k_{ij}\underline{\sigma}_k\Big) \\ &= x^j a^i \varepsilon^k_{ij} \underline{\sigma}_k = x^j \Big(a^k \varepsilon^i_{kj}\Big)\underline{\sigma}_i \\ &=: x^j \varphi(\underline{A})^i_j \underline{\sigma}_i. \end{aligned}$$

So the isomorphism between $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ is given by

$$\varphi(\underline{A})^i_j = -\varepsilon^i_{jk} a^k : \mathfrak{su}(2) \to \mathfrak{so}(3),$$

which maps the basis $\underline{s}_k$ of $\mathfrak{su}(2)$ onto the basis o_k of $\mathfrak{so}(3)$ given above. With this knowledge, we can say that because Lie algebras are locally identical to their corresponding Lie groups, SU(2) and SO(3) are locally isomorphic.