Quantum Field Theory II – Assignment 4

Exercise sheet 4, Advanced QFT, Prof. Jürgen Berges

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Problem 4.1 (The 1PI effective action)

In the lecture, we have defined

$$\tau(x_1, ..., x_n) = \frac{\delta}{i\delta J(x_1)} ... \frac{\delta}{i\delta J(x_n)} iW[J], \tag{1}$$

$$\tilde{\Gamma}(x_1, ..., x_n) = \frac{\delta}{\delta\varphi(x_1)} ... \frac{\delta}{\delta\varphi(x_n)} i\Gamma[\varphi]. \tag{2}$$

What is the interpretation of $\tau(x_1, ..., x_n)$ and $\Gamma(x_1, ..., x_n)$? We have presented both a computational and a graphic method to relate $\tau(x_1, x_2, x_3)$ and $\Gamma(x_1, x_2, x_3)$. Revise these methods and use them to relate $\tau(x_1, x_2, x_3, x_4)$ and $\Gamma(x_1, x_2, x_3, x_4)$. Argue that the result demonstrates that $\Gamma[\varphi]$ is indeed a generating functional of the 1PI-correlators.

In analogy to $G_J(x_1, ..., x_n) = \prod_{j=1}^n \frac{\delta}{i\delta J(x_j)} \frac{Z[J]}{Z[0]}$, where $\frac{Z[J]}{Z[0]}$ is the generating functional for all n-point Green's functions $G_J(x_1, ..., x_n)$ in the presence of the source J, $\tau(x_1, ..., x_n) = G_J^{(c)}(x_1, ..., x_n) = \prod_{j=1}^n \frac{\delta}{i\delta J(x_j)} iW[J]$ denotes a fully connected n-point Green's function in the presence of J and $W[J]$ is the generating functional of connected Green's functions. The fully connected are a subset of all Green's functions. An even stronger reduction are the fully connected, amputated 1PI Green's functions $\tilde{\Gamma}(x_1, ..., x_n)$ in the presence of the field φ (or without: $\Gamma(x_1, ..., x_n) = \tilde{\Gamma}(x_1, ..., x_n)|_{\varphi=0}$) generated by the 1PI effective action $\Gamma[\varphi]$.

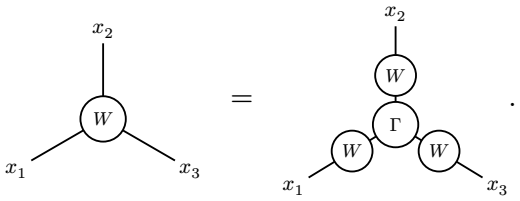
In the case of $n = 3$, $\tau(x_1, x_2, x_3)$ and $\tilde{\Gamma}(y_1, y_2, y_3)$ are related as follows,

$$\tau(x_1, x_2, x_3) = \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, \mathrm{d}^4y_3\; \tau(x_1, y_1)\tau(x_2, y_2)\tau(x_3, y_3)\tilde{\Gamma}(y_1, y_2, y_3),$$

or displayed graphically

$$\tau_3(x_1, x_2, x_3) = \text{(W)}\,|_3 = x_1 \text{—(W)—} x_2 \text{—(W)—(Γ)—(W)—} x_3$$

or illustrated in diagrams,



For $n = 4$, we need to consider

$$\tau(x_1, x_2, x_3, x_4) = \frac{\delta}{i\delta J(x_4)} \tau(x_1, x_2, x_3) = \frac{\delta}{i\delta J(x_4)} \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, \mathrm{d}^4y_3\; \tau(x_1, y_1)\tau(x_2, y_2)\tau(x_3, y_3)\tilde{\Gamma}(y_1, y_2, y_3).$$

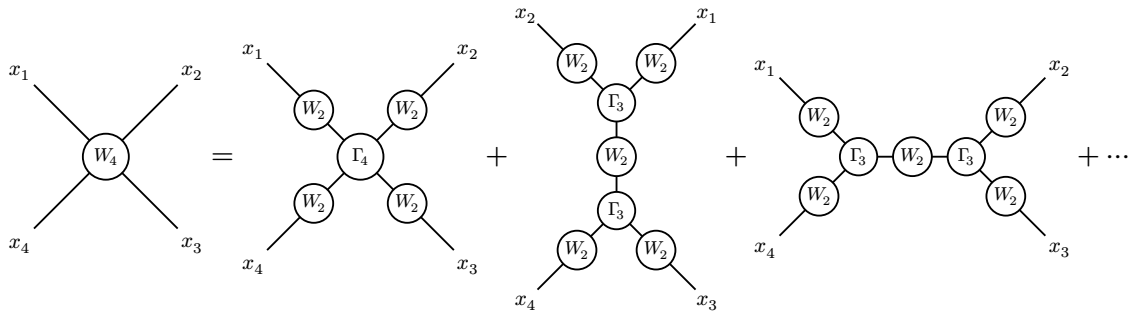
By means of the chain rule we can rewrite the functional derivative w.r.t. $iJ(x_4)$ as one w.r.t. $\varphi(y_4)$,

$$\frac{\delta}{i\delta J(x_4)} = \int \mathrm{d}^4 y_4 \, \frac{\delta \overbrace{\varphi(y_4)}^{=\frac{\delta W[J]}{\delta J(y_4)}}}{i\delta J(x_4)} \frac{\delta}{\delta\varphi(y_4)} = \int \mathrm{d}^4 y_4 \, \underbrace{\frac{\delta^2 W[J]}{i\delta J(x_4) i\delta J(y_4)}}_{\tau_2(x_4,y_4)} \frac{\delta}{\delta\varphi(y_4)} = \int \mathrm{d}^4 y_4 \, \tau_2(x_4,y_4) \frac{\delta}{\delta\varphi(y_4)}.$$

Applying the product rule, we get four terms:

$$\begin{aligned}
\tau(x_1,x_2,x_3,x_4) = &\int \prod_{k=1}^4 \mathrm{d}^4 y_k \, \tau(x_k,y_k) \underbrace{\frac{\delta}{\delta\varphi(y_4)} \tilde{\Gamma}(y_1,y_2,y_3)}_{\tilde{\Gamma}(y_1,y_2,y_3,y_4)} \\
&+ \int \prod_{k=1}^4 \mathrm{d}^4 y_k \, \tau(x_1,y_1,y_4)\tau(x_2,y_2)\tau(x_3,y_3)\tilde{\Gamma}(y_1,y_2,y_3) \\
&+ \int \prod_{k=1}^4 \mathrm{d}^4 y_k \, \tau(x_1,y_1)\tau(x_2,y_2,y_4)\tau(x_3,y_3)\tilde{\Gamma}(y_1,y_2,y_3) \\
&+ \int \prod_{k=1}^4 \mathrm{d}^4 y_k \, \tau(x_1,y_1)\tau(x_2,y_2)\tau(x_3,y_3,y_4)\tilde{\Gamma}(y_1,y_2,y_3).
\end{aligned}$$

To make better sense of the above, we may insert the expression for the case $n = 3$, i.e. $\tau(x_1, x_2, x_3)$. Illustrated again in terms of diagrams, we found that



This result was to be expected since it confirms for the case $n = 4$ that $\tilde{\Gamma}_4$ gives indeed the amputated, fully connected 1PI 4-point Green's function. The legs merely carry corrections W_2 due to the fully (Dyson) resummed propagator $G_2 = \text{—(W)—}\,|_{J=0} = \text{—●—}$, and the three additional terms are one-particle reducible (1PR) diagrams that can be split with a single cut at the central (W2). Therefore, the fourth derivative of $i\Gamma[\varphi]$ w.r.t. φ is exactly the 1PI 4-point function.

Problem 4.2 (Coleman-Weinberg potential for φ^4-theory)

In this exercise, we calculate the quantum effective action $\Gamma[\varphi]$ for the φ^4-theory,

$$S[\varphi] = \int \mathrm{d}^4 x \left(-\frac{1}{2}\varphi(x)(\partial_x^2 + \tilde{m}^2)\varphi(x) - \frac{\lambda}{4!}\varphi^4(x) \right), \quad \tilde{m}^2 = m^2 - i\varepsilon \tag{3}$$

up to one-loop order for a constant background field $\varphi = \varphi_0$. In the lecture, it was shown that

$$\Gamma[\varphi] = S[\varphi] + \hbar\gamma^{\text{1-loop}}[\varphi] + \text{'higher loop corrections'}, \quad \text{with} \tag{4}$$

$$\gamma^{\text{1-loop}}[\varphi] = -i \ln \int \hat{\mathcal{D}} f \, e^{\frac{1}{2} f \cdot \left(-\frac{i}{\hbar} \frac{\delta^2 S[\varphi]}{\delta\varphi\delta\varphi} \right) \cdot f}, \tag{5}$$

where the measure $\hat{\mathcal{D}}$ is defined such that $Z[0] = 1$.

a) Show that for any Hermitian matrix A, $\ln(\det A) = \mathrm{Tr}(\ln A)$. *Hint:* You may find it useful to perform a basis change for which A becomes diagonal.

Any complex normal matrix (satisfying $A^\dagger A = AA^\dagger$) has an orthogonal eigenvector basis. Assembling these eigenvectors into the unitary matrix U ($UU^\dagger = U^\dagger U = \mathbb{1}$), any complex normal matrix A can be decomposed

$$A = UDU^\dagger,$$

where D is a diagonal matrix which, if A is also hermitian ($A = A^\dagger$), has only real values, namely the eigenvalues λ_i of A. Performing such a change of basis, we get

$$\begin{aligned}\ln(\det A) &= \ln\big(\det\big(UDU^\dagger\big)\big) = \ln\big(\det U \cdot \det D \cdot \det U^\dagger\big) \\ &= \ln\left(\det\big(\underbrace{UU^\dagger}_{\mathbb{1}}\big)\prod_i \lambda_i\right) = \sum_i \ln(\lambda_i) = \mathrm{Tr}(\ln A).\end{aligned}$$

b) Evaluate $\gamma^{\text{1-loop}}$ by means of the path integral techniques presented in assignments 2 and 3. In analogy to part a), use that as for a differential operator D, $\ln(\det D) = \mathrm{Tr}(\ln D)$. How do you interpret the trace? *Hint:* The result can be found in the lecture notes.

$$\gamma^{\text{1-loop}}[\varphi] = -i\ln\int \hat{\mathcal{D}}f\, e^{\frac{1}{2}f\cdot\left(-\frac{i}{\hbar}\frac{\delta^2 S[\varphi]}{\delta\varphi\delta\varphi}\right)\cdot f},$$

where $\int\hat{\mathcal{D}}\varphi := \frac{1}{Z[0]}\int\mathcal{D}\varphi = \frac{1}{Z_0[0]+\mathcal{O}(\varphi^4)}\int\mathcal{D}\varphi$. Since we are performing here a perturbative expansion in powers of $\hbar$, where $\gamma^{\text{1-loop}}[\varphi]$ corresponds to order $\hbar^1$ whereas $\mathcal{O}(\varphi^4)$ contains higher powers of $\hbar$, we drop the latter and only carry along for normalization the generating functional $Z_0[0]$, of the free theory at zero source J, given by

$$Z_0[0] = \int\mathcal{D}\varphi\, e^{iS_0[\varphi]}, \quad \text{with } S_0[\varphi] = S[\varphi]|_{\lambda=0}.$$

Before performing the path integral, we explicitly calculate the exponent

$$\begin{aligned}\frac{-i}{2\hbar}\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)} &= \frac{-i}{2\hbar}\underbrace{\frac{\delta}{\delta\varphi(x)}\frac{\delta}{\delta\varphi(y)}\int \mathrm{d}^4x'\left(-\frac{1}{2}\varphi(x')(\partial_{x'}^2+\tilde{m}^2)\varphi(x')\right)}_{-(\partial_y^2+\tilde{m}^2)\delta^{(4)}(x-y),\text{ see eq. (5), assignment 2}} - \frac{\delta}{\delta\varphi(x)}\frac{\delta}{\delta\varphi(y)}\int\mathrm{d}^4x'\,\frac{\lambda}{4!}\varphi^4(x') \\ &= \frac{i}{2\hbar}\left[(\partial_x^2+\tilde{m}^2)\delta^{(4)}(x-y) + \frac{\lambda}{2}\varphi^2(x)\delta^{(4)}(x-y)\right] =: D.\end{aligned}$$

Now, inserting D back into our expression for $\gamma^{\text{1-loop}}[\varphi]$ and using our result from exercise 3.2 d), more specifically

$$\int\mathcal{D}\varphi\, e^{-\varphi\cdot D\cdot\varphi} = \sqrt{\frac{\pi^n}{\det D}}, \quad \text{for some differential operator } D,$$

where n is simply the dimensionality of the integral measure, we can carry out the path integration for a static background field $\varphi = \varphi_0$ to arrive at

$$\gamma^{\text{1-loop}}[\varphi_0] = -i\ln\frac{\int\mathcal{D}\varphi\, e^{-\varphi\cdot D\cdot\varphi}}{\int\mathcal{D}\varphi\, e^{-\varphi\cdot D_{\lambda=0}\cdot\varphi}}\bigg|_{\varphi=\varphi_0} = -i\ln\frac{\sqrt{\frac{\pi^n}{\det D}}}{\sqrt{\frac{\pi^n}{\det D_{\lambda=0}}}}\bigg|_{\varphi=\varphi_0} = -\frac{i}{2}\ln\left(\frac{\det D_{\lambda=0}}{\det D}\right)\bigg|_{\varphi=\varphi_0}.$$

Using now the expression we showed in part a), we can simplify further

$$\gamma^{\text{1-loop}}[\varphi_0] = -\frac{i}{2}\ln\left(\frac{\det D_{\lambda=0}}{\det D}\right)\Big|_{\varphi=\varphi_0} = -\frac{i}{2}\Big(\ln(\det D_{\lambda=0}) - \ln\big(\det D_{\varphi=\varphi_0}\big)\Big)$$
$$= -\frac{i}{2}\Big(\mathrm{Tr}(\ln D_{\lambda=0}) - \mathrm{Tr}\big(\ln D_{\varphi=\varphi_0}\big)\Big) = -\frac{i}{2}\ \mathrm{Tr}\left(\ln\left(\frac{D_{\lambda=0}}{D_{\varphi=\varphi_0}}\right)\right)$$
$$= -\frac{i}{2}\ \mathrm{Tr}\left(\ln\left(\frac{\frac{i}{2\hbar}(\partial_x^2+\tilde{m}^2)\delta(x-y)}{\frac{i}{2\hbar}(\partial_x^2+\tilde{m}^2+\frac{\lambda}{2}\varphi_0^2)\delta(x-y)}\right)\right) = -\frac{i}{2}\ \mathrm{Tr}\left(\ln\left(\frac{\partial_x^2+\tilde{m}^2}{\partial_x^2+\tilde{m}^2+\frac{\lambda}{2}\varphi_0^2}\right)\right).$$

The trace of a differential operator is given by the sum of its eigenvalues, for example a common operator trace appearing in statistical physics is

$$Z = \mathrm{Tr}\ e^{-\beta H} = \sum_n \langle n \mid e^{-\beta H} \mid n\rangle = \sum_n \langle n \mid e^{-\beta E_n} \mid n\rangle = \sum_n e^{-\beta E_n}.$$

A suitable eigenbasis for ∂_x^2 is e^{ikx}, however since D's eigenvalues are continuous rather than discrete, we need to integrate over the basis

$$\gamma^{\text{1-loop}}[\varphi_0] = \frac{i}{2}\ \mathrm{Tr}\left(\ln\left(\frac{\partial_x^2+\tilde{m}^2+\frac{\lambda}{2}\varphi_0^2}{\partial_x^2+\tilde{m}^2}\right)\right) \coloneqq \int \mathrm{d}^4x \int \mathrm{d}^4k \underbrace{\langle k \mid x\rangle}_{\frac{1}{(2\pi)^2}e^{ikx}} \frac{i}{2}\ln\left(\frac{\partial_x^2+\tilde{m}^2+\frac{\lambda}{2}\varphi_0^2}{\partial_x^2+\tilde{m}^2}\right) \underbrace{\langle x \mid k\rangle}_{\frac{1}{(2\pi)^2}e^{-ikx}}$$
$$= \frac{i}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(\frac{-k^2+\tilde{m}^2+\frac{\lambda}{2}\varphi_0^2}{-k^2+\tilde{m}^2}\right)e^{i(kx-kx)} = \frac{i}{2}\mathrm{Vol}_{\mathbb{R}^{1,3}}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(\frac{\tilde{m}^2+\frac{\lambda}{2}\varphi_0^2-k^2}{\tilde{m}^2-k^2}\right),$$

where we used the Taylor series representation of the logarithm,

$$\ln(\partial_x^2) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(-1+\partial_x^2)^n,$$

to let ∂_x^2 act directly on its eigenstates.

c) We define the effective potential $V(\varphi) = -\mathrm{Vol}_{\mathbb{R}^{1,3}}\Gamma[\varphi]$. Give the potential up to one loop-order, $V^{\text{tree}}(\varphi=\varphi_0) + V^{\text{1-loop}}(\varphi=\varphi_0)$. *Hint:* Starting from the expression found in part b), you should find an integral representation of $V^{\text{1-loop}}(\varphi=\varphi_0)$ of the form

$$\int \mathrm{d}^3k\left(\sqrt{\vec{k}^2+x+\delta}-\sqrt{\vec{k}^2+x}\right). \tag{6}$$

To this end, perform the integral over k^0 as a suitable contour integral. Furthermore, the following integral identity might be useful

$$\int_x^{x+\delta}\mathrm{d}\xi\,\frac{1}{\xi} = \ln\left(\frac{x+\delta}{x}\right).$$

Give a physical interpretation of the two terms in eq. (6).

Considering definitions given in the script, i.e. eqs. (7.198)–(7.202), we deduce the following:

$$\Gamma[\varphi] \coloneqq S[\varphi] + \hbar\gamma[\varphi], \quad \text{with } \gamma[\varphi] = \gamma^{\text{1-loop}}[\varphi] + \mathcal{O}(f^3), \quad \text{and}$$
$$\Gamma[\varphi] = -\mathrm{Vol}_{\mathbb{R}^{1,3}}V(\varphi) = -\mathrm{Vol}_{\mathbb{R}^{1,3}}\big(V^0(\varphi) + V^{\text{1-loop}}(\varphi) + ...\big)$$
$$\implies V^{\text{1-loop}}(\varphi_0) = -\frac{\hbar}{\mathrm{Vol}_{\mathbb{R}^{1,3}}}\gamma^{\text{1-loop}}[\varphi_0] = \frac{\hbar}{2i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(\frac{m^2+\frac{\lambda}{2}\varphi_0^2-k^2-i\varepsilon}{m^2-k^2-i\varepsilon}\right).$$

Using the hinted at integral identity backwards, we can rewrite the above,

$$V^{\text{1-loop}}(\varphi_0) = \frac{-\hbar}{2i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_{m^2}^{m^2+\frac{\lambda}{2}\varphi_0^2}\mathrm{d}\mu^2\,\frac{1}{k^2-\mu^2+i\varepsilon}$$
$$= \frac{-\hbar}{2i}\int_{m^2}^{m^2+\frac{\lambda}{2}\varphi_0^2}\mathrm{d}\mu^2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\int_C\frac{\mathrm{d}k^0}{2\pi}\,\frac{1}{k^0-\sqrt{\vec{k}^2+\mu^2-i\varepsilon}}\,\frac{1}{k^0+\sqrt{\vec{k}^2+\mu^2-i\varepsilon}},$$

where C is a path along the real axis closed at infinity by a half circle in the upper complex plane, thereby picking up the pole at $k^0=-\sqrt{\vec{k}^2+\mu^2-i\varepsilon}$ (just above the real axis) when applying the residue theorem to perform the k^0-integral. We pick up the contribution,

$$\operatorname{Res}\left(\frac{1}{k^0+\sqrt{\vec{k}^2+\mu^2-i\varepsilon}},-\sqrt{\vec{k}^2+\mu^2-i\varepsilon}\right)=\lim_{k^0\to-\sqrt{\vec{k}^2+\mu^2-i\varepsilon}}\left(k^0+\sqrt{\vec{k}^2+\mu^2-i\varepsilon}\right)\frac{1}{k^0+\sqrt{\vec{k}^2+\mu^2-i\varepsilon}}=1,$$

and no change in sign, because C is positively oriented, so that

$$V^{\text{1-loop}}(\varphi_0) = \frac{-\hbar}{2i}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\int_{m^2}^{m^2+\frac{\lambda}{2}\varphi_0^2}\mathrm{d}\mu^2\,\frac{1}{2\pi}2\pi i\,\frac{1}{-\sqrt{\vec{k}^2+\mu^2-i\varepsilon}-\sqrt{\vec{k}^2+\mu^2-i\varepsilon}}$$
$$\overset{\varepsilon\to0}{=}\frac{\hbar}{4}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\int_{m^2}^{m^2+\frac{\lambda}{2}\varphi_0^2}\frac{\mathrm{d}\mu^2}{\sqrt{\vec{k}^2+\mu^2}}$$
$$=\frac{\hbar}{4}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,2\sqrt{\vec{k}^2+\mu^2}\Big|_{m^2}^{m^2+\frac{\lambda}{2}\varphi_0^2}=\frac{\hbar}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left(\sqrt{\vec{k}^2+m^2+\frac{\lambda}{2}\varphi_0^2}-\sqrt{\vec{k}^2+m^2}\right).$$

The one-loop correction is the difference of the zero-point energies $\frac{1}{2}\hbar\omega_{\vec{k}}$ of all field modes in the background φ_0, where the effective mass is shifted to $m^2+\frac{\lambda}{2}\varphi_0^2$, and in the free vacuum.