Quantum Field Theory II – Assignment 6

Exercise sheet 6, Advanced QFT, Prof. Jürgen Berges

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Problem 6.1 (Path integral of the fermionic field with sources)

We define the generating functional for the correlation functions of the free Dirac theory as

$$Z_0[\overline{\eta},\eta] = \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi e^{i\left(S_0\left[\overline{\Psi},\Psi\right]+\overline{\eta}\cdot\Psi+\overline{\Psi}\cdot\eta\right)} \tag{1}$$

with $S_0\left[\overline{\Psi},\Psi\right] = \int \mathrm{d}^4x\,\overline{\Psi}(i\not{\partial} - m_0)\Psi$.

a) Show by means of completing the square that (1) can be rearranged to

$$Z_0[\overline{\eta},\eta] = Z_0[\overline{\eta}=0,\eta=0]e^{-\overline{\eta}\cdot S_F\cdot\eta}, \tag{2}$$

where S_F is the Feynman propagator of the free Dirac theory,

$$(i\not{\partial}_x - m_0 + i\varepsilon)S_F(x-y) = i\delta^{(4)}(x-y).$$

Since we are working in a free theory, we may choose to shift not the time into the complex plane via $t \to t(1-i\varepsilon)$ but instead make the replacement

$$\mathcal{H} \to \mathcal{H} - i\varepsilon\varphi^2(x), \quad \text{i.e.} \quad m_0^2 \to m_0^2 - i\varepsilon, \quad \text{(see eq. 7.80)}$$

in order to project the initial and final states to the vacuum. Thus, we may write the action as

$$S_0\left[\overline{\Psi},\Psi\right] \to \int \mathrm{d}^4x \int \mathrm{d}^4y\,\overline{\Psi}\underbrace{(i\not{\partial} - m_0 + i\varepsilon)\delta(x-y)}_{iS_F^{-1}(x-y)}\Psi = i\overline{\Psi}\cdot S_F^{-1}\cdot\Psi.$$

Completing the square yields

$$\begin{aligned} S_0\left[\overline{\Psi},\Psi\right] + \overline{\eta}\cdot\Psi + \overline{\Psi}\cdot\eta &= i\overline{\Psi}\cdot S_F^{-1}\cdot\Psi + \overline{\eta}\cdot\Psi + \overline{\Psi}\cdot\eta \\ &= \left(\overline{\Psi} - iS_F\overline{\eta}\right)\cdot iS_F^{-1}\cdot(\Psi - iS_F\eta) + i\overline{\eta}\cdot S_F\cdot\eta. \end{aligned}$$

We define $\overline{\Psi}' = \overline{\Psi} - iS_F\overline{\eta}$ and $\Psi' = \Psi - iS_F\eta$ and transform our path integration to these new variables. Since we performed just a shift, the Jacobian of the transformation is in both cases just one. Thus, the integration measure remains invariant. Altogether, this gives

$$\begin{aligned} Z_0[\overline{\eta},\eta] &= \int \mathcal{D}\overline{\Psi}'\mathcal{D}\Psi' e^{i\left(\underbrace{\overline{\Psi}'\cdot iS_F^{-1}\cdot\Psi'}_{S_0[\overline{\Psi}',\Psi']} + i\overline{\eta}\cdot S_F\cdot\eta\right)} \\ &= Z_0[\overline{\eta}=0,\eta=0]e^{-\overline{\eta}\cdot S_F\cdot\eta}. \end{aligned}$$

b) We now add an interaction part $\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\left(\overline{\Psi},\Psi\right)$ to $S_0\left[\overline{\Psi},\Psi\right]$. By proceeding as in the lecture for the bosonic case, show that the generating functional for the interacting theory can be written as

$$Z[\overline{\eta},\eta] = Z_0[\overline{\eta}=0,\eta=0]e^{\frac{\delta}{\delta\Psi}\cdot S_F\cdot\frac{\delta}{\delta\overline{\Psi}}} e^{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\left(\overline{\Psi},\Psi\right)} \cdot e^{i\overline{\eta}\cdot\Psi + i\overline{\Psi}\cdot\eta}\,|_{\overline{\Psi}=\Psi=0}. \tag{3}$$

Our full action now reads

$$S[\overline{\Psi},\Psi] = S_0[\overline{\Psi},\Psi] + S_{\text{int}}[\overline{\Psi},\Psi],$$

which makes our generating functional

$$\begin{aligned} Z[\overline{\eta},\eta] &:= \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi e^{i\left(S[\overline{\Psi},\Psi]+\overline{\eta}\cdot\Psi+\overline{\Psi}\cdot\eta\right)} \\ &= \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi e^{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\left(\overline{\Psi},\Psi\right)} e^{i\left(S_0[\overline{\Psi},\Psi]+\overline{\eta}\cdot\Psi+\overline{\Psi}\cdot\eta\right)} \\ &= e^{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\left(-\frac{\delta}{i\delta\eta},\frac{\delta}{i\delta\overline{\eta}}\right)} \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi e^{i\left(S_0[\overline{\Psi},\Psi]+\overline{\eta}\cdot\Psi+\overline{\Psi}\cdot\eta\right)}, \end{aligned}$$

where we used $F[\Psi]e^{i\overline{\eta}\cdot\Psi} = F\left[\frac{\delta}{i\delta\overline{\eta}}\right]e^{i\overline{\eta}\cdot\Psi}$ for any functional $F[\Psi]$. Inserting our result from part a), we obtain

$$\begin{aligned} Z[\overline{\eta},\eta] &= e^{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\left(-\frac{\delta}{i\delta\eta},\frac{\delta}{i\delta\overline{\eta}}\right)} Z_0[\overline{\eta},\eta] \\ &= Z_0[\overline{\eta}=0,\eta=0] e^{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\left(-\frac{\delta}{i\delta\eta},\frac{\delta}{i\delta\overline{\eta}}\right)} e^{i\overline{\eta}\cdot S_F\cdot i\eta}. \end{aligned}$$

To proceed, we need to understand how to commute two Grassmann-valued functionals containing derivatives. Note that

$$\begin{aligned} F\left[\frac{\delta}{i\delta\overline{\eta}}\right]G[\overline{\eta}]e^{i\overline{\eta}\cdot\Psi} &= F\left[\frac{\delta}{i\delta\overline{\eta}}\right]G\left[-\frac{\delta}{i\delta\Psi}\right]e^{i\overline{\eta}\cdot\Psi} \\ &= (-1)^{\deg(F)\deg(G)}G\left[-\frac{\delta}{i\delta\Psi}\right]F\left[\frac{\delta}{i\delta\overline{\eta}}\right]e^{i\overline{\eta}\cdot\Psi} = (-1)^{\deg(F)\deg(G)}G\left[-\frac{\delta}{i\delta\Psi}\right]F[\Psi]e^{i\overline{\eta}\cdot\Psi}, \end{aligned}$$

where we used eq. (7.239) to introduce a factor of $(-1)^{\deg(F)\deg(G)}$ with $\deg(F),\deg(G)\in\{0,1\}$, depending on whether F and G are Grassmann even or odd, also called bosonic and fermionic, respectively. Since we are treating fermions here, Ψ, $\overline{\Psi}$, η and $\overline{\eta}$ are all Grassmann odd. However, that makes both of the exponentials in the above expression for $Z[\overline{\eta},\eta]$ Grassmann even. Thus

$$\begin{aligned} Z[\overline{\eta},\eta] &= Z_0[\overline{\eta}=0,\eta=0] e^{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\left(-\frac{\delta}{i\delta\eta},\frac{\delta}{i\delta\overline{\eta}}\right)} e^{i\overline{\eta}\cdot S_F\cdot i\eta} \underbrace{e^{i\left(\overline{\eta}\cdot\Psi+\overline{\Psi}\cdot\eta\right)}|_{\overline{\Psi}=\Psi=0}}_{1} \\ &= Z_0[\overline{\eta}=0,\eta=0] e^{\frac{\delta}{\delta\Psi}\cdot S_F\cdot\frac{\delta}{\delta\overline{\Psi}}} e^{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\left(\overline{\Psi},\Psi\right)} e^{i\left(\overline{\eta}\cdot\Psi+\overline{\Psi}\cdot\eta\right)}|_{\overline{\Psi}=\Psi=0}. \end{aligned}$$

Problem 6.2 (Wick's theorem for fermions)

From eq. (3) one may derive that the correlation functions for the free theory are given by

$$\langle T\prod_{i=1}^{n}\Psi(x_i)\prod_{j=n+1}^{m+n}\overline{\Psi}(x_j)\rangle = e^{-\frac{\delta}{\delta\Psi}\cdot S_F\cdot\frac{\delta}{\delta\overline{\Psi}}}\prod_{i=1}^{n}\Psi(x_i)\prod_{j=n+1}^{m+n}\overline{\Psi}(x_j)|_{\overline{\Psi}=\Psi=0}. \tag{5}$$

a) Prove Wick's theorem for the free fermions, i.e. show that $\langle T\prod_{i=1}^{n}\Psi(x_i)\prod_{j=n+1}^{m+n}\overline{\Psi}(x_j)\rangle = 0$ for $n\neq m$ and

$$\begin{aligned} \langle T\prod_{i=1}^{n}\Psi(x_i)\prod_{j=n+1}^{m+n}\overline{\Psi}(x_j)\rangle = (-1)^{\frac{n}{2}(n-1)}S_F(x_1-x_{n+1})S_F(x_2-x_{n+2})\cdots S_F(x_n-x_{2n}) \\ + \text{'all other contractions with appropriate signs'}. \end{aligned} \tag{6}$$

For $n\neq m$: We can get a better grip on the correlation function in eq. (5) by writing the exponential as a series expansion,

$$\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \overline{\Psi}(x_j) \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{\delta}{\delta \Psi} \cdot S_F \cdot \frac{\delta}{\delta \overline{\Psi}} \right)^k \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \overline{\Psi}(x_j) |_{\overline{\Psi}=\Psi=0}$$
$$= \sum_{k=0}^{\min\{m,n\}} \frac{1}{k!} \left(-\frac{\delta}{\delta \Psi} \cdot S_F \cdot \frac{\delta}{\delta \overline{\Psi}} \right)^k \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{m+n} \overline{\Psi}(x_j) |_{\overline{\Psi}=\Psi=0},$$

where the sum aborts at $\min\{m,n\}$ since $m+1$ functional derivatives acting on just m fields always give zero. If further $m \neq n$, then after applying all functional derivatives, either Ψ-fields ($n > m$) or $\overline{\Psi}$-fields ($m > n$) remain in every term appearing due to the product rule. Taking $\overline{\Psi} = \Psi = 0$ in the end makes the whole correlation function vanish. Physically, this reflects charge conservation.

On the other hand, if $n = m$, we get

$$\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{2n} \overline{\Psi}(x_j) \rangle = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{\delta}{\delta \overline{\Psi}} \cdot S_F \cdot \frac{\delta}{\delta \Psi} \right)^k \prod_{i=1}^{n} \Psi(x_i) \overline{\Psi}(x_{n+i}) |_{\overline{\Psi}=\Psi=0}.$$

Again due to taking the limit $\overline{\Psi} = \Psi = 0$, in the end all terms of the sum over k disappear except for $k = n$, where we have an equal number of fields and derivatives. Thus

$$\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{2n} \overline{\Psi}(x_j) \rangle = \frac{1}{n!} \left(\frac{\delta}{\delta \overline{\Psi}} \cdot S_F \cdot \frac{\delta}{\delta \Psi} \right)^n \prod_{i=1}^{n} \Psi(x_i) \overline{\Psi}(x_{n+i})$$
$$= (-1)^{\frac{n}{2}(n-1)} S_F(x_1 - x_{n+1}) S_F(x_2 - x_{n+2}) \cdots S_F(x_n - x_{2n})$$
$$+ \text{ 'all other contractions with appropriate signs'}.$$

The last step actually requires lots of tedious combinatorics and commuting of Grassmann-odd fields and derivatives.

b) Show by means of eq. (6) that

$$\langle T \overline{\Psi}_A(x) M^A{}_B \Psi^B(x) \overline{\Psi}_C(y) \widehat{M}^C{}_D \Psi^D(y) \rangle = -\operatorname{tr}\Big(S_F(y-x) M S_F(x-y) \widehat{M} \Big)$$
$$+ \text{ 'uninteresting' } S_F(0)\text{-terms}.$$

To apply eq. (6), we note that in this case, $n = 2$ and so $(-1)^{\frac{n}{2}(n-1)} = -1$.

$$\begin{aligned}
&\langle T \overline{\Psi}_A(x) M^A{}_B \Psi^B(x) \overline{\Psi}_C(y) \widehat{M}^C{}_D \Psi^D(y) \rangle \\
&= (-1)^3 M^A{}_B \widehat{M}^C{}_D \langle T \Psi^B(x) \Psi^D(y) \overline{\Psi}_A(x) \overline{\Psi}_C(y) \rangle \\
&= (-1)^3 M^A{}_B \widehat{M}^C{}_D \Big(-S^B_{F,A}(x-x) S^D_{F,C}(y-y) + S^B_{F,C}(x-y) S^D_{F,A}(y-x) \Big) \\
&= -S^D_{F,A}(y-x) M^A{}_B S^B_{F,C}(x-y) \widehat{M}^C{}_D + S^B_{F,A}(0) M^A{}_B S^D_{F,C}(0) \widehat{M}^C{}_D \\
&= -\operatorname{tr}\Big(S_F(y-x) M S_F(x-y) \widehat{M} \Big) + \text{'}S_F(0)\text{-terms'}.
\end{aligned}$$