

General Relativity – Exercise Sheet 7

Exercise sheet 7, General Relativity, Prof. Björn Schäfer

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Problem 1 (Equation-of-state parameter meets scalar field) [15 points]

The continuity equation for the energy density of the cosmos reads

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \rho (1 + w) = 0, \quad (1)$$

with the eq.-of-state parameter $w = \frac{p}{c^2 \rho}$. For w constant, it leads to an eq. describing the time evolution of the density of the form

$$\rho = \rho_0 a^{-3(1+w)}.$$

For Λ , we had $w_\Lambda = -1$, which means ρ_Λ constant. It has been suggested that $w_\Lambda = w_\Lambda(a)$.

a) Show that

$$\rho = \rho_0 \exp \left(-3 \int_1^a d(\ln a') [1 + w(a')] \right)$$

solves eq. (1) for a scale-factor-dependent $w(a)$.

$$\dot{\rho} = \rho_0 \exp \left(-3 \int_1^a d(\ln a') [1 + w(a')] \right) \frac{d}{dt} \left(-3 \int_1^a d(\ln a') [1 + w(a')] \right),$$

where $d(\ln a') = \frac{da'}{a'} = \frac{1}{a'} \frac{da'}{dt} dt = \frac{\dot{a}'}{a'} dt$. Thus

$$\begin{aligned} \dot{\rho} &= \rho \left(-3 \int_1^a dt \frac{d}{dt} \frac{\dot{a}'}{a'} [1 + w(a')] \right) = -3\rho \left(\frac{\dot{a}}{a} [1 + w(a)] - \left[1 + \underbrace{w(1)}_{-1} \right] \right) \\ &= -3 \frac{\dot{a}}{a} \rho [1 + w(a)], \end{aligned}$$

where $w(1) = w(a(t_0)) = -1$ as stated above. Therefore,

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \rho (1 + w(a)) = \left[-3 \frac{\dot{a}}{a} (1 + w(a)) + 3 \frac{\dot{a}}{a} (1 + w(a)) \right] \rho = 0.$$

b) We have calculated that for a scalar field ϕ with a Lagrangian $L = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)$, the pressure and energy density can, under the assumption of homogeneity and isotropy of the field, be written

$$p = \frac{1}{2} \dot{\phi}^2 - V(\phi), \quad \rho c^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi). \quad (2)$$

Therefore, the eq.-of-state parameter w becomes

$$w = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)},$$

which, for a vanishing kinetic term $\dot{\phi} = 0$, mimics that of a cosmological constant, $w = -1$.

By plugging the densities and pressures of a scalar field into eq. (1), can you arrive at an e.o.m. for ϕ ?

$$\dot{\rho}c^2 = \dot{\phi}\ddot{\phi} + \frac{\partial V(\phi)}{\partial \phi} \dot{\phi} = (\ddot{\phi} + V'(\phi))\dot{\phi}.$$

Insertion into eq. (1) multiplied by c^2 yields

$$\begin{aligned} \dot{\rho}c^2 + 3\frac{\dot{a}}{a}\rho c^2(1+w) &= (\ddot{\phi} + V'(\phi))\dot{\phi} + 3\frac{\dot{a}}{a}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\left(1 + \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\right) \\ &= (\ddot{\phi} + V'(\phi))\dot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi}^2 = 0 \\ \implies 0 &= \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi). \end{aligned}$$

Since our investigation is concerned with a cosmos we assume to be isotropic and, in particular, homogeneous, we have $\phi = \phi(\vec{x}, t)$ and thus $\nabla\phi(t) = 0$.

Except for the term $3\frac{\dot{a}}{a}\dot{\phi}$, our result therefore markedly resembles the Klein-Gordon eq.

$$\square\phi + \frac{\partial V(\phi)}{\partial \phi} = 0$$

of a scalar field in some potential $V(\phi)$.

Problem 2 (Field equations) [15 points]

Show that the Einstein field equations,

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \quad (3)$$

could equally well be written as

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right] + \Lambda g_{\mu\nu}. \quad (4)$$

By contracting all indices, eq. (3) becomes (we use $g_\mu^\mu = g^{\mu\nu}g_{\mu\nu} = \delta_\mu^\mu = 4$)

$$R - \frac{R}{2}4 + 4\Lambda = -R + 4\Lambda = -\frac{8\pi G}{c^4}T,$$

where $R = R_\mu^\mu = g^{\mu\nu}R_{\mu\nu}$ and $T = T_\mu^\mu = g^{\mu\nu}T_{\mu\nu}$. We multiply our intermediate result with $-\frac{1}{2}g_{\mu\nu}$ and add it to the field eqs. (3) to get

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{R}{2}g_{\mu\nu} - 2\Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right),$$

which is equivalent to

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right) + \Lambda g_{\mu\nu}.$$

Problem 3 (A Hubble diagram for the 21st century) [10 points]

a) We cannot distinguish between cosmic and Doppler redshift. How do we know that most receding objects follow the cosmic flow and are not actively moving away?

Good question.

Problem 4 (Extra: CPT symmetry) [5 points]

Are the Einstein field equations CPT-invariant? I.e. do physical processes stay the same if one were to transform $x^\mu \rightarrow -x^\mu$ and $q \rightarrow -q$?

Yes, both sides of

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) + g_{\mu\nu} \Lambda$$

are CPT-even. This implies that an antimatter energy-momentum tensor generates a gravitational field, i.e. spacetime curvature, in the same way as one for matter does.