String Theory – Solution to Assignment 7

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1 Primary fields and radial quantization

A primary field $\varphi(z,\bar{z})$ is a tensor field under conformal transformations $z \to z'$, $\bar{z} \to \bar{z}'$ in the sense that

$$\varphi(z,\bar{z}) \to \varphi'(z',\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \varphi(z,\bar{z}), \tag{1}$$

where z, $\bar{z}$ denote arbitrary holomorphic functions.

a) How does a primary field with conformal weights h, $\bar{h}$ transform under dilations $z \to e^{\lambda} z$ and rotations $z \to e^{i\theta} z$ with $\lambda, \theta \in \mathbb{R}$? Give a physical interpretation for $h + \bar{h}$ and $h - \bar{h}$.

b) Consider the infinitesimal conformal transformation

$$z'(z) = z + \zeta(z), \qquad \bar{z}'(\bar{z}) = \bar{z} + \bar{\zeta}(\bar{z}). \tag{2}$$

Show that the infinitesimal transformation of a primary field is given by

$$\delta_{\zeta,\bar{\zeta}} \varphi(z,\bar{z}) = \left[h\partial_z \zeta + \bar{h}\partial_{\bar{z}}\bar{\zeta} + \zeta\partial_z + \bar{\zeta}\partial_{\bar{z}}\right]\varphi(z,\bar{z}). \tag{3}$$

c) On the cylinder with coordinates (τ, σ) we defined

$$\xi^{\pm} = \tau \pm \sigma = -i(\tau' \pm i\sigma), \qquad \tau = -i\tau', \tag{4}$$

and τ' is the time in Euclidean signature (after Wick rotation). After relabelling $\tau' \to \tau$ we have

$$\xi^{+} = -i\bar{\omega}, \qquad \xi^{-} = -i\omega, \qquad \omega = \tau - i\sigma. \tag{5}$$

Argue that the conformal map from the cylinder to the Riemann sphere, defined by

$$\omega \to z(\omega) = e^{\frac{2\pi}{l}\omega}, \tag{6}$$

maps left-/right-moving fields to holomorphic/antiholomorphic fields and that for a primary field $\Phi = \Phi_L(\xi^-) + \Phi_R(\xi^+)$ a mode expansion

$$\Phi_L(\xi^-) = \left(\frac{2\pi}{l}\right)^{h} \sum_n \varphi_n e^{i\frac{2\pi}{l} n\xi^-}, \qquad \Phi_R(\xi^+) = \left(\frac{2\pi}{l}\right)^{\bar{h}} \sum_n \tilde{\varphi}_n e^{i\frac{2\pi}{l} n\xi^+}, \tag{7}$$

translates into $\Phi(z,\bar{z}) = \Phi(z) + \bar{\Phi}(\bar{z})$,

$$\Phi(z) = \sum_n \varphi_n z^{-n-h}, \qquad \bar{\Phi}(\bar{z}) = \sum_n \tilde{\varphi}_n \bar{z}^{-n-\bar{h}}. \tag{8}$$

d) Argue that a time-ordered product of fields on the cylinder maps to a radially ordered product

$$\mathcal{R}[\Phi_1(z_1)\Phi_2(z_2)] = \begin{cases} \Phi_1(z_1)\Phi_2(z_2) & \text{for } |z_1| > |z_2|, \\ \Phi_2(z_2)\Phi_1(z_1) & \text{for } |z_2| > |z_1|. \end{cases} \tag{9}$$

a) Under dilations (rescalings) $z \to z' = e^{\lambda} z$, $\lambda \in \mathbb{R}$, a primary field $\varphi(z,\bar{z})$ with conformal weights h, $\bar{h}$ transforms as

$$\varphi(z,\bar{z}) \to \varphi'(z',\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}}\varphi(z,\bar{z}) = e^{-h\lambda}e^{-\bar{h}\lambda}\varphi(z,\bar{z}) = e^{-(h+\bar{h})\lambda}\varphi(z,\bar{z}). \quad (10)$$

Under rotations $z \to z' = e^{i\theta}z$, $\theta \in \mathbb{R}$, $\varphi(z,\bar{z})$ becomes

$$\varphi(z,\bar{z}) \to \varphi'(z',\bar{z}') = e^{-ih\theta}e^{i\bar{h}\theta}\varphi(z,\bar{z}) = e^{-i(h-\bar{h})\theta}\varphi(z,\bar{z}). \quad (11)$$

From eqs. (10) and (11), we infer that the sum of conformal weights $h+\bar{h} = \Delta$ gives the **scaling dimension** of a primary field $\varphi(z,\bar{z})$, whereas the difference $h-\bar{h} = s$ is the field's (conformal) **spin**.

b) Equation (1) can equivalently be written as

$$\varphi(z,\bar{z}) \to \varphi'(z,\bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^{-h}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}}\varphi(z',\bar{z}'). \quad (12)$$

For the conformal transformations $z'(z) = z + \zeta(z)$, $\bar{z}'(\bar{z}) = \bar{z} + \bar{\zeta}(\bar{z})$, we can expand each term in eq. (12) to first order in the infinitesimal (anti-)chiral fields $\zeta(z)$, $\bar{\zeta}(\bar{z})$ to get

$$\begin{aligned}
\left(\frac{\partial z'}{\partial z}\right)^{-h} &= \left(1 + \frac{\partial\zeta(z)}{\partial z}\right)^{-h} = 1 + h\partial_z\zeta(z) + \mathcal{O}(\zeta^2),\\
\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} &= \left(1 + \frac{\partial\bar{\zeta}(\bar{z})}{\partial \bar{z}}\right)^{-\bar{h}} = 1 + \bar{h}\partial_{\bar{z}}\bar{\zeta}(\bar{z}) + \mathcal{O}\big(\bar{\zeta}^2\big),\\
\varphi(z',\bar{z}') &= \varphi(z,\bar{z}) + \zeta(z)\partial_z\varphi(z,\bar{z}) + \bar{\zeta}(\bar{z})\partial_{\bar{z}}\varphi(z,\bar{z}) + \mathcal{O}\big(\zeta^2,\bar{\zeta}^2\big).
\end{aligned} \quad (13)$$

Dropping all terms beyond linear order gives the infinitesimal transformation

$$\begin{aligned}
\delta_{\zeta,\bar{\zeta}}\varphi(z,\bar{z}) &\equiv \varphi'(z,\bar{z}) - \varphi(z,\bar{z})\\
&= (1 + h\partial_z\zeta(z))\big(1 + \bar{h}\partial_{\bar{z}}\bar{\zeta}(\bar{z})\big)\big(1 + \zeta(z)\partial_z + \bar{\zeta}(\bar{z})\partial_{\bar{z}}\big)\varphi(z,\bar{z}) - \varphi(z,\bar{z})\\
&= \big[1 + h\partial_z\zeta(z) + \bar{h}\partial_{\bar{z}}\bar{\zeta}(\bar{z}) + \zeta(z)\partial_z + \bar{\zeta}(\bar{z})\partial_{\bar{z}}\big]\varphi(z,\bar{z}) - \varphi(z,\bar{z})\\
&= \big[h\partial_z\zeta(z) + \bar{h}\partial_{\bar{z}}\bar{\zeta}(\bar{z}) + \zeta(z)\partial_z + \bar{\zeta}(\bar{z})\partial_{\bar{z}}\big]\varphi(z,\bar{z}).
\end{aligned} \quad (14)$$

c) A Wick rotation on the cylinder with coordinates $(\tau,\sigma) \in \mathbb{R}^2$ sends

$$\xi^{\pm} = \tau \pm \sigma \xrightarrow{\text{W.R.}} -i(\tau \pm i\sigma) \equiv \begin{cases} -i\bar{\omega} = \xi^+, \\ -i\omega = \xi^-, \end{cases} \quad (15)$$

where now $\tau \in i\mathbb{R}$ and ω, $\bar{\omega}$ are coordinates on the cylinder.
After Wick rotating, the conformal map

$$i\xi^- = \omega \to z(\omega) = e^{\frac{2\pi}{l}\omega}, \qquad i\xi^+ = \bar{\omega} \to \bar{z}(\bar{\omega}) = e^{\frac{2\pi}{l}\bar{\omega}}. \quad (16)$$

from the cylinder to the Riemann sphere respectively the complex plane transforms left-movers into holomorphic fields $\Phi_L(\xi^-) \to \Phi(z(\omega))$ and right-movers into antiholomorphic ones $\Phi_R(\xi^+) \to \Phi(\bar{z}(\bar{\omega}))$.

This can be illustrated for the case of a primary field $\Phi(\xi^+,\xi^-) = \Phi_L(\xi^-) + \Phi_R(\xi^+)$ with conformal weights $\big(h,\bar{h}\big)$, where the left- and right-moving components are given by

$$\Phi_L(\xi^-) = \left(\frac{2\pi}{l}\right)^h \sum_n \varphi_n e^{i\frac{2\pi}{l}n\xi^-}, \qquad \Phi_R(\xi^+) = \left(\frac{2\pi}{l}\right)^{\bar{h}} \sum_n \tilde{\varphi}_n e^{i\frac{2\pi}{l}n\xi^+}. \quad (17)$$

Under (16), these transform into

$$
\begin{aligned}
\Phi_L(\xi^-) \to \Phi_L(z) &= \left(\frac{\partial z}{\partial \omega}\right)^{-h} \Phi_L(\xi^-) = \left(\frac{2\pi}{l} e^{\frac{2\pi}{l}\omega}\right)^{-h} \left(\frac{2\pi}{l}\right)^{h} \sum_n \varphi_n e^{\frac{2\pi}{l} n\omega} \\
&= \sum_n \varphi_n \left(e^{\frac{2\pi}{l}\omega}\right)^{n-h} = \sum_n \varphi_n z^{n-h} = \sum_n \varphi_{-n} z^{-n-h}, \\
\Phi_R(\xi^+) \to \Phi_R(\bar{z}) &= \left(\frac{\partial \bar{z}}{\partial \bar{\omega}}\right)^{-\bar{h}} \Phi_R(\xi^+) = \left(\frac{2\pi}{l} e^{\frac{2\pi}{l}\bar{\omega}}\right)^{-\bar{h}} \left(\frac{2\pi}{l}\right)^{\bar{h}} \sum_n \tilde{\varphi}_n e^{\frac{2\pi}{l} n\bar{\omega}} \\
&= \sum_n \tilde{\varphi}_n \left(e^{\frac{2\pi}{l}\bar{\omega}}\right)^{n-\bar{h}} = \sum_n \tilde{\varphi}_n \bar{z}^{n-\bar{h}} = \sum_n \tilde{\varphi}_{-n} \bar{z}^{-n-\bar{h}}.
\end{aligned}
\tag{18}
$$

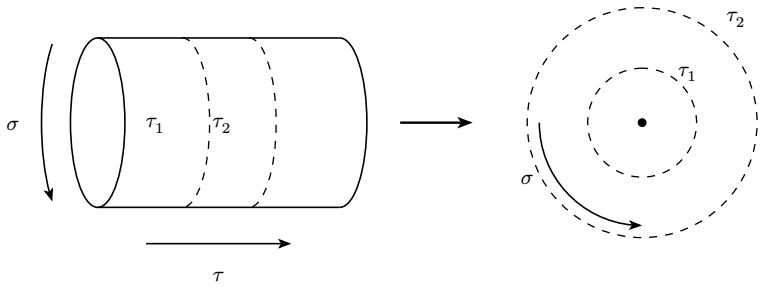
Since $i\frac{2\pi}{l} n\xi^- = \frac{2\pi}{l} n\omega$, this reproduces the stated expansion up to relabelling the modes $\varphi_n \to \varphi_{-n}$ (equivalently, it matches exactly if the cylinder modes are defined with $e^{-i\frac{2\pi}{l} n\xi^\mp}$ instead).

d) To see that time ordering on the cylinder corresponds to radial ordering on the complex plane, we choose any two points $\omega_1 = \tau_1 - i\sigma_1$, $\omega_2 = \tau_2 - i\sigma_2$ such that $\tau_1 \leq \tau_2$, i.e. in a time-ordered operator product containing τ_1 and τ_2, the operator evaluated at τ_1 would act first. For the corresponding points z_1, z_2 on the plane, we have the relation

$$
\begin{aligned}
\tau_1 \leq \tau_2 \qquad \Longleftrightarrow \qquad |z_1| &= |e^{\frac{2\pi}{l}\omega_1}| = |e^{\frac{2\pi}{l}(\tau_1 - i\sigma_1)}| = e^{\frac{2\pi}{l}\tau_1} \\
&\leq e^{\frac{2\pi}{l}\tau_2} = |e^{\frac{2\pi}{l}(\tau_2 - i\sigma_2)}| = |z_2|.
\end{aligned}
\tag{19}
$$

Thus in a radially ordered product, an operator evaluated on the plane at z_1 would also act first.

The following graphic visualizes how different times on the cylinder correspond to circles of different radii on the plane.



2 The OPE of the energy momentum tensor

In the complex plane, the Virasoro generators L_n, $n \in \mathbb{Z}$ are given by

$$
L_n = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z). \tag{20}
$$

a) Show that

$$
[L_m, L_n] = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} z^{m+1} w^{n+1} \mathcal{R}[T(z)T(w)], \tag{21}
$$

where C_0 denotes a contour about $z = 0$ and C_w is a contour about $z = w$. As usual the product $\mathcal{R}[T(z)T(w)]$ is meant to be the radially ordered product.

Hint: Write the commutator as a difference of two double contour integrals and use a contour deformation of the $\mathrm{d}z$ integration for fixed w.

b) Use eq. (21) and the (radially ordered) operator product

$$\mathcal{R}[T(z)T(w)] = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \underbrace{\mathcal{O}[(z-w)^0]}_{\text{finite terms}}, \tag{22}$$

as well as Cauchy's integral formula,

$$\oint_{C_w} \frac{\mathrm{d}z'}{2\pi i} \frac{f(z')}{(z'-z)^n} = \frac{1}{(n-1)!} f^{(n-1)}(z), \tag{23}$$

to rederive the quantum Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n}. \tag{24}$$

c) The Schwarzian derivative is defined as

$$S(\tilde{z}, z) = \frac{\partial^3 \tilde{z}}{\partial^3 z}\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 \tilde{z}}{\partial^2 z}\right)^2 \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2}. \tag{25}$$

Show that the transformation

$$T(z) \to \tilde{T}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \Big[T(z) - \frac{c}{12} S(\tilde{z}, z)\Big], \tag{26}$$

gives at the infinitesimal level for $\tilde{z} = z + \epsilon(z)$,

$$\delta T(z) = \epsilon(z)\partial T(z) + 2[\partial_z \epsilon(z)]T(z) + \frac{c}{12}\partial_z^3 \epsilon(z). \tag{27}$$

d) Apply this to the map from the cylinder to the plane, given by

$$\tau - i\sigma \equiv \omega \to z(\omega) = e^{\frac{2\pi}{l}\omega}, \tag{28}$$

to show that

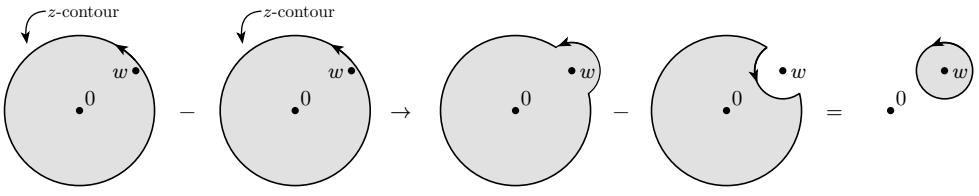
$$T_{\text{cyl}}(\omega) = \left(\frac{2\pi}{l}\right)^2 \Big(z^2 T_{\text{pln}}(z) - \frac{c}{24}\Big). \tag{29}$$

Which are the various physical interpretations for c?

a) Expanding the commutator of Virasoro generators gives

$$\begin{aligned}[L_m, L_n] &= \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{m+1} T(z) \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} w^{n+1} T(w) - \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} w^{n+1} T(w) \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{m+1} T(z) \\ &= \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \left(\oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{m+1} w^{n+1} T(z)T(w) - \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{m+1} w^{n+1} T(w)T(z) \right).\end{aligned} \tag{30}$$

We use the following contour deformation for the two integrals in parenthesis so that *after* the deformation, we have $|w| \leq |z|\ \forall z$ in the first integral and $|w| \geq |z|\ \forall z$ in the second.



Then eq. (30) simplifies to

$$[L_m, L_n] = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} z^{m+1} w^{n+1} \mathcal{R}[T(z)T(w)]. \tag{31}$$

b) To rederive[1] the quantum Virasoro algebra, we insert eq. (22) into eq. (31) to get

$$[L_m, L_n] = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} z^{m+1} w^{n+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \right] \tag{32}$$

We can perform the z-integration using Cauchy's integral formula

$$\oint_{C_w} \frac{\mathrm{d}z'}{2\pi i} \frac{f(z')}{(z'-z)^n} = \frac{1}{(n-1)!} f^{(n-1)}(z). \tag{33}$$

The three terms in brackets thus become

$$\begin{aligned} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{\frac{c}{2} z^{m+1} w^{n+1}}{(z-w)^4} &= \frac{c}{2 \cdot 3!} (m+1)m(m-1) w^{m-2} w^{n+1} = \frac{c}{12} m(m^2-1) w^{m+n-1}, \\ \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{2T(w) z^{m+1} w^{n+1}}{(z-w)^2} &= 2T(w)(m+1) w^m w^{n+1}, \\ \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{\partial_w T(w) z^{m+1} w^{n+1}}{z-w} &= \partial_w T(w) w^{m+1} w^{n+1}. \end{aligned} \tag{34}$$

We reinsert eq. (34) into eq. (32) and get

$$\begin{aligned} [L_m, L_n] &= \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \left[\frac{c}{12} m(m^2-1) w^{m+n-1} + 2T(w)(m+1) w^m w^{n+1} + \partial_w T(w) w^{m+1} w^{n+1} \right] \\ &= \frac{c}{12} m(m^2-1) \delta_{m,-n} + \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} [\underbrace{T(w)(m+1) w^m w^{n+1}}_{(m+1)L_{m+n}} + \partial_w [T(w) w^{m+1}] w^{n+1}] \end{aligned} \tag{35}$$

Integration by parts in the last term where the boundary terms cancel since we integrate over the closed contour C_0 gives

$$\begin{aligned} \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \partial_w [T(w) w^{m+1}] w^{n+1} &= - \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} T(w) w^{m+1} \partial_w w^{n+1} \\ &= -(n+1) \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} T(w) w^{m+1} w^n = -(n+1) L_{m+n}. \end{aligned} \tag{36}$$

Thus we arrive at the renowned Virasoro algebra

$$\begin{aligned} [L_m, L_n] &= \frac{c}{12} m(m^2-1) \delta_{m,-n} + (m+1) L_{m+n} - (n+1) L_{m+n} \\ &= (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m,-n}. \end{aligned} \tag{37}$$

> **Note:** We have not *really* rederived the Virasoro algebra here. Instead the operator product expansion (22) was really constructed in such a way as to reproduce this known commutator of two Virasoro generators.

c) The change of chiral fields (in a two-dimensional quantum field theory) under infinitesimal conformal transformations can be calculated using the conformal Ward-Takahashi identity. For transformations of the form $\tilde{z} = z + \epsilon(z)$ it reads

[1] See exercise 1 on assignment 4 for the first (painstakingly long) derivation.

$$\delta_\epsilon \mathcal{O}(z) = \oint_{C_z} \frac{\mathrm{d}z'}{2\pi i}\epsilon(z')\mathcal{R}[T(z')\mathcal{O}(z)]. \tag{38}$$

Since we are interested in the transformational behavior of the energy-momentum tensor, we can again use the operator product expansion

$$\mathcal{R}[T(z')T(z)] = \frac{c/2}{(z'-z)^4} + \frac{2T(z)}{(z'-z)^2} + \frac{\partial_z T(z)}{z'-z} + \underbrace{\mathcal{O}\big[(z'-z)^0\big]}_{\text{finite terms}}, \tag{39}$$

as well as Cauchy's integral formula (23) to compute

$$\begin{aligned}\delta_\epsilon T(z) &= \oint_{C_z} \frac{\mathrm{d}z'}{2\pi i}\epsilon(z')\mathcal{R}[T(z')T(z)] = \oint_{C_z} \frac{\mathrm{d}z'}{2\pi i}\epsilon(z')\left[\frac{\partial_z T(z)}{z'-z} + \frac{2T(z)}{(z'-z)^2} + \frac{c/2}{(z'-z)^4}\right] \\ &= \epsilon(z)\partial_z T(z) + 2[\partial_z \epsilon(z)]T(z) + \frac{c}{12}\partial_z^3\epsilon(z),\end{aligned} \tag{40}$$

which is indeed the transformation we were asked to derive. Equation (40) confirms that $T(z)$ transforms as a tensor of weight $(2,0)$ but only for special transformations that satisfy $\partial_z^3\epsilon(z) = 0$ or for general transformations if $c = 0$.

> **Note:** Instead of doing all the expansions by hand in section 1, part b), we could equally well have used the Ward-Takahashi identity there as well to calculate the change of primary fields under conformal transformations.

d) The Schwarzian derivative of the map $\omega \to z(\omega) = e^{\frac{2\pi}{l}\omega}$ from the cylinder to the plane reads

$$\begin{aligned}S(z,\omega) &= \frac{\partial^3 z}{\partial^3\omega}\left(\frac{\partial z}{\partial\omega}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 z}{\partial^2\omega}\right)^2\left(\frac{\partial z}{\partial\omega}\right)^{-2} \\ &= \left(\frac{2\pi}{l}\right)^3 z\left(\frac{2\pi}{l}\right)^{-1} z^{-1} - \frac{3}{2}\left(\frac{2\pi}{l}\right)^4 z^2\left(\frac{2\pi}{l}\right)^{-2} z^{-2} = -\frac{1}{2}\left(\frac{2\pi}{l}\right)^2\end{aligned} \tag{41}$$

Solving eq. (26) for the energy-momentum tensor on the cylinder yields

$$T_{\text{pln}}(z) = \left(\frac{\partial z}{\partial\omega}\right)^{-2}\Big[T_{\text{cyl}}(\omega) - \frac{c}{12}S(z,\omega)\Big] \quad \Rightarrow \quad T_{\text{cyl}}(\omega) = \left(\frac{\partial z}{\partial\omega}\right)^2 T_{\text{pln}}(z) + \frac{c}{12}S(z,\omega). \tag{42}$$

We plug in our result for the Schwarzian derivative and get

$$T_{\text{cyl}}(\omega) = \left(\frac{2\pi}{l}\right)^2 z^2 T_{\text{pln}}(z) - \frac{1}{2}\left(\frac{2\pi}{l}\right)^2\frac{c}{12} = \left(\frac{2\pi}{l}\right)^2\Big(z^2 T_{\text{pln}}(z) - \frac{c}{24}\Big). \tag{43}$$

We interpret the central term c of the quantum Virasoro algebra as the Casimir energy.

> **Note:** Operating on the plane means dealing with flat infinitely extended spacetime. Intuitively, the lowest possible energy, i.e. the zero-point energy $\langle H_0\rangle$, on this space should vanish (see lecture notes, p. 81). Since Hamiltonian and energy-momentum tensor are directly related, $H \propto \int \mathrm{d}\sigma T(\tau,\sigma)$, this particularly implies that the one-point function on the plane must also vanish, i.e. $\big\langle T_{\text{pln}}(z)\big\rangle = 0$. Using this information in eq. (43), we get the very interesting result
>
> $$\big\langle T_{\text{cyl}}(\omega)\big\rangle = -\frac{c}{24}\left(\frac{2\pi}{l}\right)^2. \tag{44}$$

Not only does eq. (44) demonstrate, that the one-point function on the cylinder is non-vanishing, it also seems to introduce a length scale into our theory, thereby breaking the conformal symmetry it enjoyed thus far. This is due to the underlying geometry containing a length scale: the circumference l of the cylinder.

3 Bonus question: Fractional linear transformations

The purpose of this exercise is to show that conformal transformations on the Riemann sphere (i.e. on $\mathbb{C} \cup \{\infty\}$) map any 3 points to any other 3 points. The conformal group on the Riemann-sphere is given by $SL(2,\mathbb{C})$ of 2×2-matrices with unit determinant acting on the Riemann-sphere by so-called fractional linear transformations

$$z \to z' = \frac{az+b}{cz+d}, \tag{45}$$

where

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}). \tag{46}$$

a) Show that two successive fractional linear transformations,

$$z \to z' = \frac{az+b}{cz+d}, \qquad z' \to z'' = \frac{ez'+f}{gz'+h}, \tag{47}$$

are equivalent to one fractional linear transformation

$$z \to z'' = \frac{jz+k}{lz+m}, \tag{48}$$

where the matrix

$$\begin{pmatrix} j & k \\ l & m \end{pmatrix} \in SL(2,\mathbb{C}) \tag{49}$$

is the product of the two $SL(2,\mathbb{C})$ matrices that correspond to the single transformations $z \to z'$ and $z' \to z''$.

b) Show that the fractional linear action of the inverse matrix of (46) on z' leads back to z, and hence corresponds to the inverse transformation $z' \to z$.

c) Consider the map

$$z \to z' = \frac{(b-c)(z-a)}{(b-a)(z-c)}. \tag{50}$$

Show that this defines, up to an overall rescaling, an $SL(2,\mathbb{C})$ transformation provided a, b, c are pairwise distinct. Use this to show that $SL(2,\mathbb{C})$ maps any 3 distinct points on S^2 to any other 3 distinct points.

d) Show that the cross-ratio

$$\langle z_1, z_2, z_3, z_4 \rangle \equiv \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} \tag{51}$$

is $SL(2,\mathbb{C})$ invariant. Use this to show that

$$\langle z, a, b, c \rangle = z', \tag{52}$$

where z' is the one given in part c).

a) The conformal group on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is the group of special linear transformations of degree 2,

$$SL(2,\mathbb{C}) = \{\boldsymbol{A} \in \mathrm{Mat}(2\times 2,\mathbb{C}) \,|\, \det(\boldsymbol{A}) = 1\}, \tag{53}$$

with ordinary matrix multiplication and matrix inversion as group operations. $SL(2,\mathbb{C})$ can be parametrized by fractional-linear transformations (also known as Möbius transformations)

$$z \to z' = \frac{az+b}{cz+d}, \qquad \boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}). \tag{54}$$

Performing two successive fractional linear transformations, $z \to z' = \frac{az+b}{cz+d}$ followed by $z' \to z'' = \frac{ez'+f}{gz'+h}$, we get

$$z'' = \frac{e\frac{az+b}{cz+d}+f}{g\frac{az+b}{cz+d}+h} = \frac{aez+be+cfz+df}{agz+bg+chz+dh} = \frac{(ae+cf)z+(be+df)}{(ag+ch)z+(bg+dh)} \equiv \frac{jz+k}{lz+m}. \tag{55}$$

Thus, the overall transformation is given by

$$\boldsymbol{C} = \begin{pmatrix} j & k \\ l & m \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}. \tag{56}$$

Since

$$\begin{aligned} \det(\boldsymbol{C}) &= (ae+cf)(bg+dh) - (ag+ch)(be+df) \\ &= \underline{abeg} + aedh + cfbg + \underline{\underline{cfdh}} - \underline{agbe} - agdf - chbe - \underline{\underline{chdf}} \\ &= aedh + cfbg - agdf - chbe = ad\underbrace{(eh-gf)}_{1} + bc\underbrace{(fg-eh)}_{-1} = ad - bc = 1, \quad \checkmark \end{aligned} \tag{57}$$

$\boldsymbol{C}$ is also in $SL(2,\mathbb{C})$. Hence, any two successive $SL(2,\mathbb{C})$-transformations give a third. This is not surprising as we had already asserted $SL(2,\mathbb{C})$ to be a group.

b) The inverse of $\boldsymbol{A}$ from eq. (46) is given by

$$\boldsymbol{A}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{58}$$

Applying the fractional linear transformation corresponding to $\boldsymbol{A}^{-1}$ to z' returns z,

$$\frac{dz'-b}{-cz'+a} = \frac{d\frac{az+b}{cz+d}-b}{-c\frac{az+b}{cz+d}+a} = \frac{daz+\underline{db}-bcz-\underline{bd}}{\underline{\underline{-acz}}-cb+\underline{\underline{acz}}+ad} = \frac{(ad-bc)z}{ad-cb} = z. \quad \checkmark \tag{59}$$

The inverse of transformation (45) is indeed represented by $\boldsymbol{A}^{-1}$.

c) Reshaped into the standard form of a fractional linear transformation, eq. (50) reads

$$z \to z' = \frac{(b-c)z - a(b-c)}{(b-a)z - c(b-a)}. \tag{60}$$

The corresponding $SL(2,\mathbb{C})$-matrix

$$\boldsymbol{D} = \begin{pmatrix} b-c & (c-b)a \\ b-a & (a-b)c \end{pmatrix} \tag{61}$$

has determinant

$$\det(\boldsymbol{D}) = (b-c)(a-b)c - (b-a)(c-b)a = (a-b)(b-c)(c-a). \tag{62}$$

As long as $a \neq b \wedge a \neq c \wedge b \neq c$, $\boldsymbol{D}$ has non-vanishing determinant and can be made $SL(2,\mathbb{C})$ with a simple scale factor of $\boldsymbol{D}' = \boldsymbol{D}/\sqrt{\det(\boldsymbol{D})}$.

As for showing that $SL(2,\mathbb{C})$ maps any 3 distinct points on S^2 to any other 3 distinct points, we note that the transformation (50) maps

$$z' = \frac{(b-c)(z-a)}{(b-a)(z-c)} = \begin{cases} 0 & \text{for } z = a, \\ 1 & \text{for } z = b, \\ \infty & \text{for } z = c, \end{cases} \tag{63}$$

where $\{0, 1, \infty\}$ all lie on S^2 and we already showed that a, b, and c are arbitrary but distinct points on S^2. Thus, all that remains to be shown is that there exists an $SL(2,\mathbb{C})$-transformation that maps $\{0, 1, \infty\}$ to any three distinct points $\{z_1, z_2, z_3\}$ on S^2. Fortunately, we already know that a transformation of the form (50), i.e.

$$z \to z' = \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}, \qquad \text{with } \begin{pmatrix} z_2 - z_3 & (z_3 - z_2)z_1 \\ z_2 - z_1 & (z_1 - z_2)z_3 \end{pmatrix} \equiv \boldsymbol{E} \tag{64}$$

maps $\{z_1, z_2, z_3\}$ to $\{0, 1, \infty\}$ and can be made $SL(2,\mathbb{C})$ by scaling

$$SL(2,\mathbb{C}) \quad \ni \quad \boldsymbol{E}' = \frac{1}{\sqrt{\det(\boldsymbol{E})}}\,\boldsymbol{E} = \frac{1}{\sqrt{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}}\,\boldsymbol{E}. \tag{65}$$

Thus the combined transformation $\boldsymbol{F} \equiv \boldsymbol{E}'^{-1}\boldsymbol{D}'$ maps any three points $\{a, b, c\}$ to any other three points $\{z_1, z_2, z_3\}$ on S^2, where the points in each set are pairwise distinct.

d) A simple calculation shows that the cross-ratio defined in eq. (51) is invariant under $SL(2,\mathbb{C})$-transformations,

$$\begin{aligned} \langle z_1, z_2, z_3, z_4\rangle \to \langle z_1', z_2', z_3', z_4'\rangle = \frac{(z_1' - z_2')(z_3' - z_4')}{(z_1' - z_4')(z_3' - z_2')} &\overset{(50)}{=} \frac{\left(\frac{z_1-a}{z_1-c} - \frac{z_2-a}{z_2-c}\right)\left(\frac{z_3-a}{z_3-c} - \frac{z_4-a}{z_4-c}\right)}{\left(\frac{z_1-a}{z_1-c} - \frac{z_4-a}{z_4-c}\right)\left(\frac{z_3-a}{z_3-c} - \frac{z_2-a}{z_2-c}\right)} \\ &= \frac{\frac{(z_1-z_2)(z_3-z_4)(a-c)^2}{(z_1-c)(z_2-c)(z_3-c)(z_4-c)}}{\frac{(z_1-z_4)(z_3-z_2)(a-c)^2}{(z_1-c)(z_2-c)(z_3-c)(z_4-c)}} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \langle z_1, z_2, z_3, z_4\rangle. \end{aligned} \tag{66}$$

Completely unrelated and simply by definition, we have

$$\langle z, a, b, c\rangle \overset{(51)}{=} \frac{(z-a)(b-c)}{(z-c)(b-a)} \overset{(50)}{=} z'. \tag{67}$$