General Relativity – Exercise Sheet 7

Exercise sheet 7, General Relativity, Prof. Björn Schäfer

Solutions by Janosh Riebesell, 19.07.2015

Problem 1 (Equation-of-state parameter meets scalar field) [15 points]

The continuity equation for the energy density of the cosmos reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w) = 0, \tag{1}$$

with the eq.-of-state parameter $w = \frac{p}{c^2\rho}$. For w constant, it leads to an eq. describing the time evolution of the density of the form

$$\rho = \rho_0 a^{-3(1+w)}.$$

For Λ, we had $w_\Lambda = -1$, which means ρ_Λ constant. It has been suggested that $w_\Lambda = w_\Lambda(a)$.

a) Show that

$$\rho = \rho_0 \exp\left(-3\int_1^a \mathrm{d}(\ln a')[1+w(a')]\right)$$

solves eq. (1) for a scale-factor-dependent $w(a)$.

$$\dot{\rho} = \rho_0 \exp\left(-3\int_1^a \mathrm{d}(\ln a')[1+w(a')]\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(-3\int_1^a \frac{\mathrm{d}a'}{a'}[1+w(a')]\right).$$

Since the lower integration limit is constant, the fundamental theorem of calculus and the chain rule give

$$\dot{\rho} = \rho\left(-3\frac{[1+w(a)]}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right) = -3\frac{\dot{a}}{a}\rho[1+w(a)].$$

Therefore,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w(a)) = \left[-3\frac{\dot{a}}{a}(1+w(a)) + 3\frac{\dot{a}}{a}(1+w(a))\right]\rho = 0.$$

b) We have calculated that for a scalar field ϕ with a Lagrangian $L = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$, the pressure and energy density can, under the assumption of homogeneity and isotropy of the field, be written

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad \rho c^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{2}$$

Therefore, the eq.-of-state parameter w becomes

$$w = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)},$$

which, for a vanishing kinetic term $\dot{\phi} = 0$, mimics that of a cosmological constant, $w = -1$.

By plugging the densities and pressures of a scalar field into eq. (1), can you arrive at an e.o.m. for ϕ?

$$\dot{\rho}c^2 = \dot{\phi}\ddot{\phi} + \frac{\partial V(\phi)}{\partial \phi}\dot{\phi} = \Big(\ddot{\phi} + V'(\phi)\Big)\dot{\phi}.$$

Insertion into eq. (1) multiplied by c^2 yields

$$\begin{aligned}
\dot{\rho}c^2 + 3\frac{\dot{a}}{a}\rho c^2(1+w) &= \Big(\ddot{\phi} + V'(\phi)\Big)\dot{\phi} + 3\frac{\dot{a}}{a}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\left(1 + \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\right) \\
&= \Big(\ddot{\phi} + V'(\phi)\Big)\dot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi}^2 = 0 \\
\implies \quad 0 &= \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi).
\end{aligned}$$

Since our investigation is concerned with a cosmos we assume to be isotropic and, in particular, homogeneous, we have $\phi = \phi(\cancel{x}, t)$ and thus $\nabla\phi(t) = 0$.

Except for the term $3\frac{\dot{a}}{a}\dot{\phi}$, our result therefore markedly resembles the Klein-Gordon eq.

$$\Box\phi + \frac{\partial V(\phi)}{\partial \phi} = 0$$

of a scalar field in some potential $V(\phi)$.

Problem 2 (Field equations) [15 points]

Show that the Einstein field equations,

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{3}$$

could equally well be written as

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right] + \Lambda g_{\mu\nu}. \tag{4}$$

By contracting all indices, eq. (3) becomes (we use $g^\mu_\mu = g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4$)

$$R - \frac{R}{2}4 + 4\Lambda = -R + 4\Lambda = -\frac{8\pi G}{c^4}T,$$

where $R = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu}$ and $T = T^\mu_\mu = g^{\mu\nu}T_{\mu\nu}$. We multiply our intermediate result with $-\frac{1}{2}g_{\mu\nu}$ and add it to the field eqs. (3) to get

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{R}{2}g_{\mu\nu} - 2\Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right),$$

which is equivalent to

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right) + \Lambda g_{\mu\nu}.$$

Problem 3 (A Hubble diagram for the 21st century) [10 points]

a) We cannot distinguish between cosmic and Doppler redshift. How do we know that most receding objects follow the cosmic flow and are not actively moving away?

Through Hubble's law: redshifts grow linearly with independently measured distances (e.g. from Cepheids or type Ia supernovae) in all directions on the sky. Peculiar velocities, on the other hand, are randomly oriented and only of order a few hundred km/s, so beyond a few tens of Mpc they make up only a small scatter around the cosmic flow.

Problem 4 (Extra: CPT symmetry) [5 points]

Are the Einstein field equations CPT-invariant? I.e. do physical processes stay the same if one were to transform $x^\mu \to -x^\mu$ and $q \to -q$?

Yes, both sides of

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\,g_{\mu\nu}\right) + g_{\mu\nu}\Lambda$$

are CPT-even. This implies that an antimatter energy-momentum tensor generates a gravitational field, i.e. spacetime curvature, in the same way as one for matter does.