Quantum Field Theory II – Assignment 8

Exercise sheet 8, Advanced QFT, Prof. Jürgen Berges

Solutions by Janosh Riebesell, 07.06.2015

Problem 8.1 (Superficial degree of divergence for QED)

Review the Feynman rules of quantum electrodynamics and derive the superficial degree of divergence D for this theory. Proceed similarly to the derivation of D for the φ^4-theory as it was presented in the lecture.

The superficial degree of divergence D is given by the power of momenta in the highest order, i.e. most divergent, contribution to an amplitude. Every loop yields a momentum integral $\int \mathrm{d}^4 k_i$ and therefore $d = 4$ powers of momentum. All other powers derive from the propagators of internal lines. To specify these, we recall the QED Feynman rules for propagators

$$\text{internal fermion line:} \quad \xrightarrow{\quad p \quad} \;\hat{=}\; \frac{i(\not{p} + m_0)}{p^2 - m_0^2} = \frac{i(\not{p} + m_0)}{(\not{p} + m_0)(\not{p} - m_0)} = \frac{i}{\not{p} - m_0},$$

$$\text{internal photon line:} \quad \mu \sim\sim\sim\sim\sim\sim \nu \;\hat{=}\; -\frac{i\eta_{\mu\nu}}{q^2},$$

$$\left(\text{vertex:} \quad \begin{array}{l} p_2 \searrow \\ \qquad\quad \sim\sim\sim\sim\sim\sim \mu \\ p_1 \nearrow \end{array} \;\hat{=}\; -ie\gamma^\mu \delta^{(4)}(p_1 - p_2 - q)\right).$$

We expect D to be a function of any of the following

$$D = D\big(E_f, I_f, E_\gamma, I_\gamma, V, L\big),$$

where the arguments are the numbers of

$$\begin{aligned} &E_f : \text{external fermions}, && I_f : \text{internal fermions}, \\ &E_\gamma : \text{external photons}, && I_\gamma : \text{internal photons}, \\ &V : \text{QED vertices}, && L : \text{QED loops}. \end{aligned}$$

From the Feynman rules, we easily see the UV-structure of QED amplitudes is given by

$$A_{\mathrm{UV}} \sim \int \frac{\prod_{l=1}^{L} \mathrm{d}^4 k_l}{\prod_{i=1}^{I_f} (\not{p}_i - m_0) \cdot \prod_{j=1}^{I_\gamma} q_j^2},$$

resulting in a superficial degree of divergence of

$$D = 4L - I_f - 2I_\gamma.$$

We proceed by inserting for the number of loop integrations

$$L = I_f + I_\gamma - (V - 1),$$

which again follows from the Feynman rules as they attribute one momentum integral to each internal line $I = I_f + I_\gamma$ while each vertex contributes one $\delta^{(4)}$, one of which only enforces overall momentum conservation. Thus

$$D = 4\big(I_f + I_\gamma - (V - 1)\big) - I_f - 2I_\gamma = 4 - 4V + 3I_f + 2I_\gamma.$$

The number of vertices can also be reexpressed by

$$V = 2I_\gamma + E_\gamma = \frac{1}{2}\big(2I_f + E_f\big),$$

since every QED vertex has one photon- and two fermion-lines, and every internal line is connected to two vertices. Altogether, we get

$$\begin{aligned} D &= 4 - \big(2I_\gamma + E_\gamma\big) - 3 \cdot \frac{1}{2}\big(2I_f + E_f\big) + 3I_f + 2I_\gamma \\ &= 4 - E_\gamma - \frac{3}{2}E_f. \end{aligned}$$

Additional note: Looking at $\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma\cdot\partial - m_0)\psi - eA_\mu\overline{\psi}\gamma^\mu\psi$, we see that since $[\psi] = \frac{3}{2}$, $[A_\mu] = 1$, QED's coupling has mass-dimension $[e] = 0$. From this, we conclude that QED is renormalizable in 4 dimensions, i.e. has finitely many divergencies, but at every order in perturbation theory.

Problem 8.2 (One-loop structure of QED)

From problem 8.1, you should have found that the superficial degree of divergence of a QED-amplitude with an interaction-Lagrangian $\mathcal{L}_{\text{int}} = \overline{\psi}\gamma^\mu A_\mu\psi$ is

$$D = 4 - E_\gamma - \frac{3}{2}E_f. \tag{1}$$

Since $[e] = 0$, the theory is renormalizable.

a) Give D for each of the following amplitudes.

i) The vacuum energy.



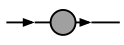
$E_f = E_\gamma = 0, \qquad D = 4 - E_\gamma - \frac{3}{2}E_f = 4$

ii) The photon propagator.



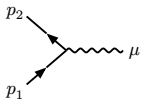
$E_f = 0, E_\gamma = 2, \quad D = 4 - E_\gamma = 2$

iii) The electron propagator.



$E_f = 2, E_\gamma = 0, \quad D = 4 - \frac{3}{2}E_f = 1$

iv) The electron-electron-photon vertex.



$E_f = 2, E_\gamma = 1, \quad D = 4 - E_\gamma - \frac{3}{2}E_f = 0$

v) The 1-photon amplitude.



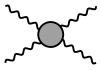
$E_f = 0, E_\gamma = 1, \quad D = 4 - E_\gamma = 3$

vi) The 3-photon amplitude.



$$E_f = 0, E_\gamma = 3, \quad D = 4 - E_\gamma = 1$$

vii) The 4-photon amplitude.



$$E_f = 0, E_\gamma = 4, \quad D = 4 - E_\gamma = 0$$

b) Verify explicitly that the one-loop diagram contributing to the one-point function vanishes using the discrete symmetry of charge conjugation, $j^\mu \to -j^\mu$, $A^\mu \to -A^\mu$. There are two Feynman diagrams contributing to the three-point function at one-loop order. Show that these cancel. Show that the diagrams contributing to any n-point photon amplitude, for n odd, cancel in pairs.

Charge conjugation $\hat{C}$ is a symmetry of QED, meaning

$$\hat{C}\ |\Omega\rangle = |\Omega\rangle.$$

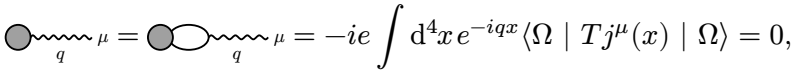
Since $j^\mu(x)$ changes sign under charge conjugation,

$$\hat{C} j^\mu(x) \hat{C}^\dagger = -j^\mu(x), \quad \text{where } \hat{C}^\dagger \hat{C} = \mathrm{id},$$

we infer that its vacuum expectation value vanishes at one-loop (and any other number of loops),

$$\begin{aligned}\langle \Omega \mid T j^\mu(x) \mid \Omega \rangle &= \langle \Omega \mid j^\mu(x) \mid \Omega \rangle = \langle \Omega \mid \hat{C}^\dagger \hat{C} j^\mu(x) \hat{C}^\dagger \hat{C} \mid \Omega \rangle \\ &= -\langle \Omega \mid j^\mu(x) \mid \Omega \rangle = 0.\end{aligned}$$

This carries over to the photon one-point function



$$= -ie \int \mathrm{d}^4x\, e^{-iqx} \langle \Omega \mid T j^\mu(x) \mid \Omega \rangle = 0,$$

where $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$.

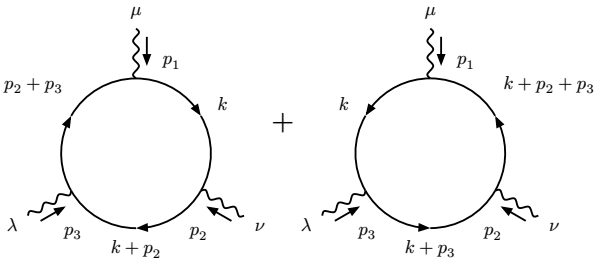
More explicitly, we may also write the one-point amplitude at one-loop as

$$\Gamma^{(1)\mu}|_{\text{1-loop}} = -ie \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{-i\ \mathrm{tr}[\gamma^\mu(\not{k} + m)]}{k^2 - m^2} = -e \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{4k^\mu}{k^2 - m^2} = 0,$$

where we used $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$ and $\mathrm{tr}(\gamma^\mu) = 0$ (see pr. 9.3, QFT I), and which vanishes, since k^μ is odd and $\frac{1}{k^2-m^2}$ is even.

The three-point amplitude at one-loop consists of the two diagrams

$$i\Gamma^{(3)\mu\nu\lambda}|_{\text{1-loop}} =$$



$$= -(-ie)^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left\{ \mathrm{tr}[\gamma^\mu S_F(k) \gamma^\nu S_F(k + p_2) \gamma^\lambda S_F(k + p_2 + p_3)] \right.$$

$$+ \operatorname{tr}[\gamma^\mu S_F(-k)\gamma^\nu S_F(-k-p_2)\gamma^\lambda S_F(-k-p_2-p_3)]\},$$

where $S_F(k) \coloneqq \frac{1}{\not{k}-m} = \frac{\not{k}+m}{k^2-m^2}$. To show that this vanishes, we use γ^μ's and S_F's behavior under charge conjugation,

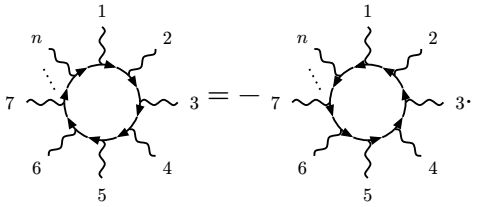
$$\hat{C}^\dagger \gamma^\mu \hat{C} = (\gamma^\mu)^T, \quad \hat{C}^\dagger S_F(k) \hat{C} = -S_F(-k),$$

and the cyclicity of the trace (to send the last $\hat{C}^\dagger$ to the front). Thus

$$\begin{aligned}
&\operatorname{tr}[\gamma^\mu \underbrace{\hat{C}\hat{C}^\dagger}_{\mathrm{id}} S_F(-k)\hat{C}\hat{C}^\dagger\gamma^\nu\hat{C}\hat{C}^\dagger S_F(-k-p_2)\hat{C}\hat{C}^\dagger\gamma^\lambda\hat{C}\hat{C}^\dagger S_F(-k-p_2-p_3)\hat{C}\hat{C}^\dagger] \\
&= \operatorname{tr}\Big[(\gamma^\mu)^T(-S_F(k))(\gamma^\nu)^T(-S_F(k+p_2))\big(\gamma^\lambda\big)^T(-S_F(k+p_2+p_3))\Big] \\
&= -\operatorname{tr}[\gamma^\mu S_F(k)\gamma^\nu S_F(k+p_2)\gamma^\lambda S_F(k+p_2+p_3)].
\end{aligned}$$

Reinserting into $i\Gamma^{(3)}|_{\text{1-loop}}$ yields immediately $i\Gamma^{(3)}|_{\text{1-loop}} = 0$.

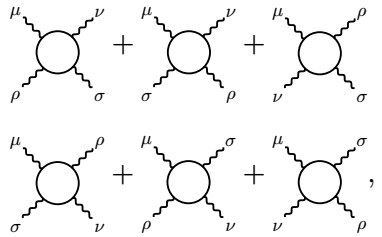
We turn to the general case of all diagrams contributing to a photon n-point function where n is odd. First note, that every fermionic loop with an odd number of vertices has a counterpart in which the momenta point along the loop in opposite direction



Since an n-point photon function receives contributions from $(n-1)!$ diagrams, and $(n-1)!$ is even for $n > 2$ (the case $n = 1$, i.e. the one-loop diagram , we already showed to be vanishing in the beginning) we can always arrange all contributing diagrams in pairs of two, where for the special case of n odd, we may partner each diagram with its oppositely running counterpart in order to let each pair of our partition vanish individually.

c) The photon four-point amplitude is a sum of six diagrams. Show explicitly that the potential logarithmic divergences of these diagrams cancel.

The six diagrams are

$\Gamma^{(4)\mu\nu\rho\sigma} =$ 

where we omit momentum arrows in the fermion loops since clock- and counterclockwise running are equal for an even number of vertices.

The potential log-divergence of $i\Gamma^{(4)\mu\nu\rho\sigma}$ stems entirely from integrating over the internal momentum, i.e.

$$i\Gamma^{(4)\mu\nu\rho\sigma}_{\text{div}} \sim \int \mathrm{d}^4k \, \frac{(\not{k})^4}{(k^2)^4},$$

and is the same irrespective of a change in external momenta p_i, $i \in \{1,2,3,4\}$, or, for that matter, the fermion mass m. To speed up calculations, we may therefore set both to zero.

The divergent part $i\Gamma^{(4)\mu\nu\rho\sigma}_{\text{div}}$ of the photon 4-point amplitude may then be written much more compactly as

$$\begin{aligned} i\Gamma^{(4)\mu\nu\rho\sigma}_{\text{div}} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{-1}{\left(k^2\right)^4} [&\mathrm{tr}(\gamma^\mu \not{k} \gamma^\nu \not{k} \gamma^\rho \not{k} \gamma^\sigma \not{k}) + \mathrm{tr}(\gamma^\mu \not{k} \gamma^\nu \not{k} \gamma^\sigma \not{k} \gamma^\rho \not{k}) \\ &+ \mathrm{tr}(\gamma^\mu \not{k} \gamma^\rho \not{k} \gamma^\nu \not{k} \gamma^\sigma \not{k}) + \mathrm{tr}(\gamma^\mu \not{k} \gamma^\rho \not{k} \gamma^\sigma \not{k} \gamma^\nu \not{k}) \\ &+ \mathrm{tr}(\gamma^\mu \not{k} \gamma^\sigma \not{k} \gamma^\nu \not{k} \gamma^\rho \not{k}) + \mathrm{tr}(\gamma^\mu \not{k} \gamma^\sigma \not{k} \gamma^\rho \not{k} \gamma^\nu \not{k})]. \end{aligned}$$

To get a grasp on this expression, we concern ourselves solely with the first trace initially. Using the gamma-matrices' anti-commutation relations and some of the trace identities worked out in problem 9.3 of QFT I, it can be expressed as

$$\begin{aligned} \mathrm{tr}(\gamma^\mu \not{k} \gamma^\nu \not{k} \gamma^\rho \not{k} \gamma^\sigma \not{k}) = 32k^\mu k^\nu k^\rho k^\sigma - 8k^2(&k^\mu k^\nu g^{\rho\sigma} + k^\rho k^\sigma g^{\mu\nu} \\ &+ k^\mu k^\sigma g^{\nu\rho} + k^\nu k^\rho g^{\mu\sigma}) + 4k^4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \end{aligned}$$

We symmetrize products of k by sending $k^\mu k^\nu \to \frac{k^2}{4} g^{\mu\nu}$ and $k^\mu k^\nu k^\rho k^\sigma \to \frac{k^4}{24}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$ to obtain

$$\mathrm{tr}(\gamma^\mu \not{k} \gamma^\nu \not{k} \gamma^\rho \not{k} \gamma^\sigma \not{k}) = \frac{4}{3} k^4 (g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}).$$

From this expression, the other five traces follow simply by permutations of indices. Since the metric is symmetric, they all sum up to zero.