Quantum Field Theory II – Assignment 10

Exercise sheet 10, Advanced QFT, Prof. Jürgen Berges

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Problem 10.1 (One-loop β-function in QED)

a) Compute the one-loop β-function in QED.

$$\beta_e(\mu) = \mu\frac{\partial}{\partial\mu}\Big(-e\delta_1 + e\delta_2 + \frac{e}{2}\delta_3\Big), \quad \text{where} \quad -\mu^2 = 4m^2, \text{i.e.} \quad \mu \propto m.$$

$$\delta_1 = -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z(1-z)\Bigg(\frac{\Gamma\big(2-\frac{d}{2}\big)}{\big((1-z)^2m^2+zM^2\big)^{2-d/2}}\,\frac{1}{2}(2-\varepsilon)^2 + \frac{\Gamma\big(3-\frac{d}{2}\big)}{\big((1-z)^2m^2+zM^2\big)^{3-d/2}}\big(2(1-4z+z^2)-\varepsilon(1-z)^2\big)m^2\Bigg),$$

$$\begin{aligned}\frac{\partial\delta_1}{\partial\mu} &= -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z(1-z)\Bigg[\frac{-\overbrace{\big(2-\frac{d}{2}\big)\Gamma\big(2-\frac{d}{2}\big)}^{\Gamma(3-d/2)=\Gamma(1+\varepsilon/2)=1}}{\big((1-z)^2m^2+zM^2\big)^{3-d/2}}2(1-z)^2m\,\frac{1}{2}(2-\varepsilon)^2 \\ &\qquad + \frac{\partial}{\partial\mu}\,\frac{\Gamma(1)}{(1-z)^2+zM^2/m^2}\,2(1-4z+z^2)\Bigg] \\ &\overset{\varepsilon\to0}{=} -\frac{e^2}{(4\pi)^2}\int_0^1 \mathrm{d}z(1-z)\left[\frac{-4(1-z)^2m}{(1-z)^2m^2+zM^2} + \frac{2(1-4z+z^2)}{\big((1-z)^2+zM^2/m^2\big)^2}\,z\,\frac{2M^2}{m^3}\right] \\ &\overset{M\to0}{=} -\frac{e^2}{(4\pi)^2}\int_0^1 \mathrm{d}z(1-z)\,\frac{-4(1-z)^2m}{(1-z)^2m^2} = \frac{4e^2}{16\pi^2m}\int_0^1 \mathrm{d}z(1-z) = \frac{e^2}{8\pi^2m},\end{aligned}$$

where we let $M \to 0$ as it is an IR cutoff for the photon propagator. Further, we took $e(\mu) = e$ to be constant at one-loop order.

$$\begin{aligned}\frac{\partial\delta_2}{\partial\mu} &\overset{M\to0}{=} -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,\frac{\partial}{\partial\mu}\left[\frac{\Gamma\big(2-\frac{d}{2}\big)}{\big((1-z)m\big)^{4-d}}\left((2-\varepsilon)z - \frac{\varepsilon}{2}\,\frac{z}{1-z}\,(4-2z-\varepsilon(1-z))\right)\right] \\ &= -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,\frac{-2\overbrace{\big(2-\frac{d}{2}\big)\Gamma\big(2-\frac{d}{2}\big)}^{1}}{\big((1-z)m\big)^{5-d}}\left((2-\varepsilon)z - \frac{\varepsilon}{2}\,\frac{z}{1-z}\,(4-2z-\varepsilon(1-z))\right)(1-z) \\ &= \frac{2e^2}{(4\pi)^2}\int_0^1 \mathrm{d}z\,\frac{2z}{m} = \frac{e^2}{8\pi^2m},\end{aligned}$$

$$\delta_3 = -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,\frac{\Gamma\big(2-\frac{d}{2}\big)}{(m^2)^{2-d/2}}\,(8z(1-z)),$$

$$\frac{\partial\delta_3}{\partial\mu} = -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,\frac{-2\overbrace{\big(2-\frac{d}{2}\big)\Gamma\big(2-\frac{d}{2}\big)}^{1}}{m^{5-d}}\,(8z(1-z)) = \frac{2e^2}{(4\pi)^2m}\int_0^1 \mathrm{d}z\,8z(1-z)$$

$$= \frac{e^2}{\pi^2 m}\int_0^1 \mathrm{d}z\,(z-z^2) = \frac{e^2}{6\pi^2 m}\,.$$

Thus, putting everything together, we get

$$\begin{aligned}\beta_e(\mu) &= \mu\frac{\partial}{\partial\mu}\Big(-e\delta_1 + e\delta_2 + \frac{e}{2}\delta_3\Big)\\ &= -e\frac{e^2}{8\pi^2} + e\frac{e^2}{8\pi^2} + \frac{e}{2}\frac{e^2}{6\pi^2} = \frac{e^3}{12\pi^2}\,.\end{aligned}$$

b) Integrate the solution to the renormalization group equation $\frac{\mathrm{d}}{\mathrm{d}\log\mu}e(\mu) = \beta_e(\mu)$ to find at one-loop order

$$\alpha(\mu) = \frac{\alpha^*}{1-\frac{2}{3\pi}\alpha^*\log\!\left(\frac{\mu}{\mu^*}\right)}, \quad \alpha = \frac{e^2}{4\pi},$$

where $\alpha^* = \alpha(\mu^*)$ with μ^* some fixed scale. Plot the behavior of $e(\mu)$. At which scale does e cease to be perturbative?

$$\frac{\mathrm{d}}{\mathrm{d}\log(\mu)}e(\mu) = \beta_e(\mu) \quad\Longrightarrow\quad \frac{1}{\beta_e(\mu)}\,\mathrm{d}e(\mu) = \mathrm{d}\log(\mu).$$

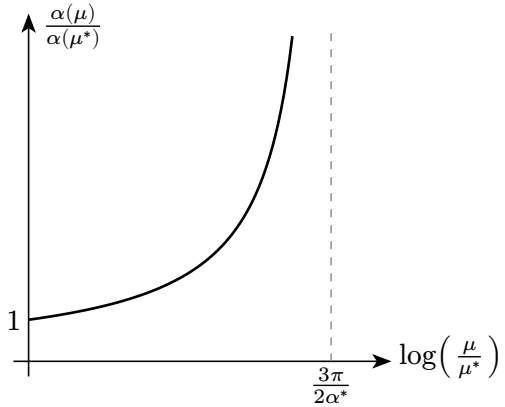
With $\alpha(\mu) = \frac{e(\mu)^2}{4\pi}$, i.e. $\mathrm{d}\alpha(\mu) = \frac{e(\mu)}{2\pi}\,\mathrm{d}e(\mu)$, the left-hand side integrates to

$$\begin{aligned}\int_{\mu^*}^{\mu}\frac{1}{\beta_e(\mu')}\,\mathrm{d}e(\mu') &= \int_{\mu^*}^{\mu}\frac{12\pi^2}{e^3(\mu')}\frac{2\pi}{e(\mu')}\,\mathrm{d}\alpha(\mu') = \int_{\mu^*}^{\mu}\frac{\frac{3}{2}\pi}{\alpha^2(\mu')}\,\mathrm{d}\alpha(\mu')\\ &= -\frac{3}{2}\pi\left(\frac{1}{\alpha(\mu)} - \frac{1}{\alpha(\mu^*)}\right),\end{aligned}$$

while the right-hand side gives

$$\int_{\mu^*}^{\mu}\mathrm{d}\log(\mu') = \log\!\left(\frac{\mu}{\mu^*}\right) \overset{!}{=} \frac{3}{2}\pi\left(\frac{1}{\alpha(\mu^*)} - \frac{1}{\alpha(\mu)}\right) \quad\Longrightarrow\quad \frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu^*)} - \frac{2}{3\pi}\log\!\left(\frac{\mu}{\mu^*}\right)$$

$$\Longrightarrow\quad \alpha(\mu) = \frac{\alpha(\mu^*)}{1-\frac{2}{3\pi}\alpha(\mu^*)\log\!\left(\frac{\mu}{\mu^*}\right)}\,.$$



Perturbativeness is lost at $e \approx 1$, i.e.

$$\frac{1}{4\pi} \approx \frac{\alpha(\mu^*)}{1-\frac{2\alpha(\mu^*)}{3\pi}\log\!\left(\frac{\mu}{\mu^*}\right)}$$

$$\Longrightarrow\quad \mu \approx \mu^*\exp\left[\frac{3}{2}\pi\left(\frac{1}{\alpha(\mu^*)} - 4\pi\right)\right].$$

Problem 10.2 (Anomalous dimension)

a) Consider a two-point function $G_2(p;\lambda,\mu)$ of a massless scalar theory and rewrite it as

$$G_2(p;\lambda,\mu) = \frac{i}{p^2} f\left(-\frac{p^2}{\mu^2}\right) \tag{4}$$

with an a priori arbitrary function f. Show that the Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + 2\gamma(\lambda)\right) G_2(p;\lambda,\mu) = 0 \tag{5}$$

can be cast into the form

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda} + 2 - 2\gamma(\lambda)\right) G_2(p;\lambda,\mu) = 0, \tag{6}$$

where $p = \sqrt{-p^2}$.

To arrive at eq. (6), we need a relation between the derivatives of $G_2(p;\lambda,\mu)$ with respect to μ and p. We compute

$$\begin{aligned}
\mu\frac{\partial}{\partial\mu} G_2(p;\lambda,\mu) &= \mu\frac{i}{p^2}\,\frac{2p^2}{\mu^3}\,\frac{\partial f(-p^2/\mu^2)}{\partial(-p^2/\mu^2)} = \frac{2i}{\mu^2} f'\left(-\frac{p^2}{\mu^2}\right), \\
p\frac{\partial}{\partial p} G_2(p;\lambda,\mu) &= p\frac{-2i}{p^3} f\left(-\frac{p^2}{\mu^2}\right) + p\frac{i}{p^2}\,\frac{-2p}{\mu^2} f'\left(-\frac{p^2}{\mu^2}\right) \\
&= -2G_2(p;\lambda,\mu) - \frac{2i}{\mu^2} f'\left(-\frac{p^2}{\mu^2}\right).
\end{aligned}$$

Thus, we found the relation $\mu\frac{\partial}{\partial\mu} G_2(p;\lambda,\mu) = -p\frac{\partial}{\partial p} G_2(p;\lambda,\mu) - 2G_2(p;\lambda,\mu)$. Insertion into eq. (5) yields

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda} + 2(1-\gamma)\right) G_2(p;\lambda,\mu) = 0.$$

b) Integrate this equation to find

$$G_2(p;\lambda,\mu) = \frac{i}{p^2}\mathcal{F}\big(\overline{\lambda}(p;\lambda)\big)\exp\left(2\int_{p'=\mu}^{p} \mathrm{d}\left[\log\frac{p'}{\mu}\right]\gamma\big(\overline{\lambda}(p';\lambda)\big)\right), \tag{7}$$

where $\overline{\lambda}(p;\lambda)$ is the running coupling, which is defined such that $p\frac{\partial}{\partial p}\overline{\lambda}(p;\lambda) = \beta\big(\overline{\lambda}(p;\lambda)\big)$ with the initial condition $\overline{\lambda}(\mu;\lambda) = \lambda$, and $\mathcal{F}(x)$ is some a priori unknown function. *Hint:* Compare eq. (6) with

$$\left(p\frac{\mathrm{d}}{\mathrm{d}p} + 2(1-\gamma(\lambda^*))\right) G_2\left(p;\overline{\lambda}\left(\frac{\mu^2}{p};\lambda^*\right),\mu\right)\Bigg|_{\lambda^*=\overline{\lambda}(p;\lambda)} = 0. \tag{8}$$

Note that $\overline{\lambda}(\mu^2/p;x)$ is the inverse of $\overline{\lambda}(p;x)$, i.e. $\overline{\lambda}\big(\frac{\mu^2}{p};\overline{\lambda}(p;x)\big) = x$.

We can easily verify that eqs. (6) and (8) are completely equivalent by noting that

$$
\begin{aligned}
p\frac{\mathrm{d}}{\mathrm{d}p}G_2\left(p;\overline{\lambda}\left(\frac{\mu^2}{p};\lambda^*\right),\mu\right)\Bigg|_{\lambda^*=\overline{\lambda}(p;\lambda)} &= p\frac{\partial}{\partial p}G_2\left(p;\overline{\lambda}\left(\frac{\mu^2}{p};\lambda^*\right),\mu\right)\Bigg|_{\lambda^*=\overline{\lambda}(p;\lambda)} \\
&\quad + p\frac{\partial}{\partial p}\overline{\lambda}\left(\frac{\mu^2}{p};\lambda^*\right)\frac{\partial}{\partial\overline{\lambda}(\mu^2/p;\lambda^*)}G_2\left(p;\overline{\lambda}\left(\frac{\mu^2}{p};\lambda^*\right),\mu\right)\Bigg|_{\lambda^*=\overline{\lambda}(p;\lambda)} \\
&= p\frac{\partial}{\partial p}G_2\left(p;\overline{\lambda}\left(\frac{\mu^2}{p};\overline{\lambda}(p;\lambda)\right),\mu\right) \\
&\quad + \underbrace{p\frac{\partial}{\partial p}\overline{\lambda}\left(\frac{\mu^2}{p};\overline{\lambda}(p;\lambda)\right)}_{\beta\left(\overline{\lambda}\left(\mu^2/p;\overline{\lambda}(p;\lambda)\right)\right)}\frac{\partial}{\partial\overline{\lambda}\left(\mu^2/p;\overline{\lambda}(p;\lambda)\right)}G_2\left(p;\overline{\lambda}\left(\frac{\mu^2}{p};\overline{\lambda}(p;\lambda)\right),\mu\right) \\
&= p\frac{\partial}{\partial p}G_2(p;\lambda,\mu)+\beta(\lambda)\frac{\partial}{\partial\lambda}G_2(p;\lambda,\mu),
\end{aligned}
$$

where we used the hint that $\overline{\lambda}\Big(\frac{\mu^2}{p};\overline{\lambda}(p;\lambda)\Big)=\lambda$.

Inserting this equality into eq. (8), we immediately reobtain eq. (6). Thus we may integrate eq. (8) and rest assured that our result for $G_2(p;\lambda,\mu)$ also fulfills eq. (6). Since it is a first-order differential equation, we may solve it by separation of variables:

$$
\int_\mu^p \frac{\mathrm{d}G\Big(p';\overline{\lambda}(\mu^2/p';\lambda^*),\mu\Big)}{G\Big(p';\overline{\lambda}(\mu^2/p';\lambda^*),\mu\Big)}\Bigg|_{\lambda^*=\overline{\lambda}(p;\lambda)} = -2(1-\gamma(\lambda^*))\int_\mu^p \mathrm{d}\ln(p')\;|_{\lambda^*=\overline{\lambda}(p;\lambda)},
$$

where the left-hand side evaluates to

$$
\ln(G(p';\lambda,\mu))\;|_\mu^p = \ln(G(p;\lambda,\mu)) - \ln\left(\frac{i}{\mu^2}f(-1)\right)
$$

$$
\implies\quad G(p;\lambda,\mu)=\frac{i}{\mu^2}f(-1)e^{2\int_\mu^p \mathrm{d}\ln(p')\gamma\left(\overline{\lambda}(p';\lambda)\right)}e^{-2\int_\mu^p \mathrm{d}\ln(p')}.
$$

Since $e^{-2\int_\mu^p \mathrm{d}\ln(p')}=\mu^2/p^2$, this is eq. (7) with $\mathcal{F}\Big(\overline{\lambda}(p;\lambda)\Big)=f(-1)$.

c) Argue that in the vicinity of a fixed point λ^* where $\beta(\lambda^*)=0$ the two-point function scales as

$$
G_2(p;\lambda^*,\mu)\propto\left(\frac{1}{p^2}\right)^{1-\gamma(\lambda^*)} \tag{9}
$$

and justify from this the term anomalous dimension for γ.

For $\beta(\lambda^*)=0$, eq. (6) reads at $\lambda=\lambda^*$

$$
\frac{\mathrm{d}G_2(p;\lambda^*,\mu)}{G_2(p;\lambda^*,\mu)} = -2(1-\gamma(\lambda^*))\frac{\mathrm{d}p}{p}.
$$

Integration gives

$$
\ln(G_2(p;\lambda^*,\mu)) = -2(1-\gamma(\lambda^*))\ln(p)+c = \ln\Big(p^{-2(1-\gamma(\lambda^*))}\Big)+c.
$$

Therefore, after exponentiation we find

$$
G_2(p;\lambda^*,\mu)=e^c e^{\ln\left(p^{-2(1-\gamma(\lambda^*))}\right)}\propto\left(\frac{1}{p^2}\right)^{1-\gamma(\lambda^*)},
$$

i.e. $G_2(p; \lambda^*, \mu)$ has non-integer – anomalous – dimension of momentum.