04.02.2015

Exam on Quantum Field Theory

written on 11.02.2009 at the Imperial College London (two hours)

Problem 1 (20 points)

Consider the classical real scalar field $\phi(x)$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - V(\phi).$$

You are given that the classical Poisson bracket satisfies

$$\{\phi(t,\vec{x}), \pi(t,\vec{y})\}_{PB} = -\{\pi(t,\vec{y}), \phi(t,\vec{x})\}_{PB} = \delta^{(3)}(\vec{x}-\vec{y}),$$

while $\{\phi(t,\vec{x}), \phi(t,\vec{y})\}_{PB} = 0$ and $\{AB, C\}_{PB} = A\{B,C\}_{PB} + B\{A,C\}_{PB}$.

i) What is the definition of the momentum density $\pi(x)$ conjugate $\phi(x)$?
What is it equal to in this case? Rewrite $\mathcal{L}$ in terms of $\dot{\phi}$ and $\vec{\nabla}\phi$ and hence show that the Hamiltonian is given by

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}\vec{\nabla}\phi\,\vec{\nabla}\phi + V(\phi)\right).$$

What is $V(\phi)$ for a free scalar field of mass m? What is the minimum value of H in this case?

$$\pi(x) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}(x)$$

$$H = \int d^3x\, \mathcal{H} := \int d^3x \left(\pi\dot{\phi} - \mathcal{L}\right) = \int d^3x \left(\dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right)$$

$$= \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right)$$

$V(\phi) = \frac{1}{2}m^2\phi^2$ and $H \geq 0$ because it depends on squares of the field and the field is real, i.e. $\phi^2 \in \mathbb{R}_+$

ii) Define $T^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - \eta^{\mu\nu}\left(\frac{1}{2}\partial_\lambda \phi \, \partial^\lambda \phi - V(\phi)\right)$ and

$$M^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu.$$

Show that $\partial_\mu T^{\mu\nu} = 0$ if ϕ satisfies the equation of motion $\partial_\mu \partial^\mu \phi + V'(\phi) = 0$ and hence that $\partial_\mu M^{\mu\nu\rho} = 0$.

$$\partial_\mu T^{\mu\nu} = \partial_\mu \partial^\mu \phi\, \partial^\nu \phi + \underline{\partial^\mu \phi\, \partial_\mu \partial^\nu \phi} - \tfrac{1}{2}\underline{\partial^\nu \partial_\lambda \phi\, \partial^\lambda \phi} - \tfrac{1}{2}\underline{\partial_\lambda \phi\, \partial^\nu \partial^\lambda \phi} + \partial^\nu V(\phi)$$

$$= \partial_\mu \partial^\mu \phi\, \partial^\nu \phi + \frac{\partial}{\partial \phi} V(\phi)\, \partial^\nu \phi = \big(\underbrace{\partial_\mu \partial^\mu \phi + V'(\phi)}_{0,\ \text{e.o.m.}}\big)\partial^\nu \phi = 0$$

$$\partial_\mu M^{\mu\nu\rho} = \underbrace{\partial_\mu T^{\mu\nu}}_{0} x^\rho + T^{\rho\nu} - \underbrace{\partial_\mu T^{\mu\rho}}_{0} x^\nu - T^{\nu\rho} = T^{\rho\nu} - T^{\rho\nu} = 0,$$

where we used that $T^{\mu\nu}$ is symmetric under exchange of μ and ν.

iii) Consider the integrals

$$Q^\mu = \int d^3x\, T^{0\mu}(t,\vec{x}), \qquad Q^{\mu\nu} = \int d^3x\, M^{0\mu\nu}(t,\vec{x}).$$

How do they depend on t? What does Q^μ represent physically? What about Q^{ij}, where $i,j \in \{1,2,3\}$? Write down the components of Q^μ and show explicitly that one is related to H.

Both quantities, Q^μ and $Q^{\mu\nu}$, are time-independent. To show this, we compute their time derivatives using the result of part ii).

$$\dot{Q}^\mu = \int d^3x\, \partial_t T^{0\mu}(t,\vec{x}) \overset{ii)}{=} -\int d^3x\, \partial_i T^{i\mu}(t,\vec{x}) = -\int_{\partial\mathbb{R}^3} dA_i\, T^{i\mu}(t,\vec{x}) = 0,$$

$$\dot{Q}^{\mu\nu} = \int d^3x\, \partial_t M^{0\mu\nu} \overset{ii)}{=} -\int d^3x\, \partial_i M^{i\mu\nu}(t,\vec{x}) = -\int_{\partial\mathbb{R}^3} dA_i\, M^{i\mu\nu}(t,\vec{x}) = 0,$$

where we assumed in both cases that the quantities vanish at spatial infinity.

Q^μ are the Noether charges, in this case corresponding to the total conserved four-momentum. We see this by computing

$$\int d^3x\, T^{00} = \int d^3x \left(\dot{\phi}^2 - \mathcal{L}\right) = \int d^3x \left(\pi \dot{\phi} - \mathcal{L}\right) = \int d^3x\, \mathcal{H} = H = Q^0$$

$$\int d^3x\, T^{0i} = \int d^3x\, \dot{\phi}\, \partial^i \phi = \int d^3x\, \pi\, \partial^i \phi = P^i = Q^i \text{ and hence } Q = (H, P^i) \text{ for } i \in \{1,2,3\}.$$

$$Q^{ij} = \int d^3x\, M^{0ij} = \int d^3x \left(T^{0i}x^j - T^{0j}x^i\right) = \int d^3x \left(\dot{\phi}\,\partial^i\phi\, x^j - \dot{\phi}\,\partial^j\phi\, x^i\right)$$

$$= \int d^3x\, \dot{\phi}\left(\partial^i\phi\, x^j - \partial^j\phi\, x^i\right)$$

So we find Q^{ij} to be the total angular momentum of the field ϕ.

iv) Show that $\{Q^\mu, \phi(x)\}_{PB} = -\partial_\mu\phi(x)$ and comment on the result. Comment briefly on what you expect for the Poisson brackets between different components of Q^μ and $Q^{\mu\nu}$.

$$\{Q^0, \phi(x)\}_{PB} = \int d^3x\, \{T^{00}, \phi\}_{PB} = \int d^3x\, \{\dot{\phi}^2 - \tfrac{1}{2}(\partial\phi)^2 + V(\phi), \phi\}_{PB}$$

$$= \tfrac{1}{2}\int d^3x\, \{\pi^2, \phi\}_{PB} = \int d^3x\, \pi\{\pi, \phi\}_{PB} = -\int d^3x\, \dot{\phi}\, \delta^{(3)}(\vec{x}-\vec{y}) = -\dot{\phi}$$

$$\{Q^i, \phi(x)\}_{PB} = \int d^3x\, \{T^{0i}, \phi\}_{PB} = -\int d^3x\, \{\dot{\phi}\,\partial^i\phi, \phi\}_{PB} = -\int d^3x\, \partial_i\phi\, \{\pi, \phi\}_{PB}$$

$$= +\int d^3x\, \partial^i\phi\, \delta^{(3)}(\vec{x}-\vec{y}) = \partial^i\phi$$

This reflects the fact that Q^μ are the generators of the translation symmetry group. Similarly $Q^{\mu\nu}$ are the generators of the Lorentz symmetry group. Collectively, $\{Q^\mu, Q^{\mu\nu}\}$ form a representation of the Poincaré group under the Poisson bracket.

Problem 2 (20 points)

Consider a free real scalar field $\phi(x)$ with conjugate momentum density $\pi(x) = \dot{\phi}(x)$. Define the operator

$$a_p = \int d^3x\, \frac{e^{-i\vec{p}\vec{x}}}{\sqrt{2E_p}}\left(E_p\,\phi(0,\vec{x}) + i\,\pi(0,\vec{x})\right), \qquad \text{where } E_p = \sqrt{\vec{p}^2 + m^2}.$$

i) The equal-time commutation relations (ETCRs) state that

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}).$$

Are the fields $\phi(x)$ and $\pi(x)$ in the Schrödinger or Heisenberg picture?

Why is this picture more natural in a relativistic theory. What are the ETCRs for $[\phi(t,\vec{x}),\phi(t,\vec{y})]$ and $[\pi(t,\vec{x}),\pi(t,\vec{y})]$?

The above are Heisenberg picture fields since in the Schrödinger picture, all time dependencies are carried exclusively by the states. Heisenberg picture operators are much more at home in the Minkowski space, also called spacetime, of relativistic physics.

The combinations of fields given above commute, i.e.

$$[\phi(t,\vec{x}),\phi(t,\vec{y})] = [\pi(t,\vec{x}),\pi(t,\vec{y})] = 0.$$

ii) Give an expression for $a_p^\dagger$. Using the ETCRs show that

$$[a_p, a_q^\dagger] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}), \qquad [a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0.$$